GE 3.3 Chg

Business Mathematics & Statistics

Marks 100

Module I

Business Mathematics

Internal Assessment: 10 marks
Semester-end Examinations: 40 marks
Total 50 marks

1 **Permutations and Combinations:** Definition, Factorial Notation, Theorems on Permutation, Permutations with repetitions, Restricted Permutations;Theorems on Combination, Basic identities, Restricted Combinations. **[8 L /8Marks]**

2 **Set Theory**: Definition of set, Presentation of sets, Different types of sets- Null set, Finite and infinite Sets, Universal set, Subset, Power set etc.; Set Operations, Law of algebra of Sets. **[8 L /8Marks]**

3 **Binomial Theorem:** Statement of the theorem for positive integral index, General term, Middle term, Simple properties of binomial coefficients. **[8 L /8Marks]**

4 **Logarithm:** Definition, Base and Index of Logarithm, General properties of Logarithm, Common Problems. **[8 L /8Marks]**

5 **Compound Interest and Annuities:**Simple AP and GP Series, Different types of interest rates, Net present value, Types of annuities, Continuous compounding, Valuation of simple loans and debentures, Problems relating to Sinking Funds. **[8 L /8Marks]**

Module II

Statistics

Internal Assessment: 10 marks
Semester-end Examinations: 40 marks
Total 50 marks

6. **Correlation and Association:**Bivariate data, Scatter diagram, Pearson's correlationcoefficient, Spearman's rank correlation, Measures of association of attributes. **[8 L /8Marks]**

7. **Regression Analysis:** Least squares method, Simple regression lines, properties of regression, Identification of regression lines. **[8 L /8Marks]**

8. **Index Numbers:** Meaning and types of index numbers, Problems of constructing index numbers, Construction of price and quantity indices, Test of adequacy, errors in index numbers, Chain base index numbers; Base shifting, Splicing, Deflating, Consumer price index and its uses. **[8 L /8Marks]**

9. **Time Series Analysis:** Causes of variation in time series data, Components of time series, additive and multiplicative models, Determination of trend by semi-average, moving average and least squares(of linear, quadratic and exponential trend) methods; Computation of seasonal Indices by simple average, ratio-to-moving average, ratio-to-trend and link relative methods; Simple forecasting through time series data. **[8 L /8Marks]**

10. Probability Theory: Meaning of probability; Different definitions of probability; Conditional probability; Compound probability; Independent events, Simple problems. **[8 L /8Marks]**

Permutation
&
Combination

Permutations and Combinations

Type	Formulas	Explanation of Variables	Example
Permutation with repetition (Use permutation formulas *when order matters* in the problem.)	$\boldsymbol{n^r}$	Where *n* is the number of things to choose from, and you choose *r* of them.	A lock has a 5 digit code. Each digit is chosen from 0-9, and a digit can be repeated. How many different codes can you have? n = 10, r = 5 10^5 = 100,000 codes
Permutation without repetition (Use permutation formulas *when order matters* in the problem.)	$\boldsymbol{\frac{n!}{(n-r)!}}$	Where *n* is the number of things to choose from, and you choose *r* of them. Sometimes you can see the following notation for the same concept: $P(n,r) = {}^nP_r = {}_nP_r = \frac{n!}{(n-r)!}$	How many ways can you order 3 out of 16 different pool balls? n = 16, r = 3 $\frac{16!}{(16-3)!} = 3{,}360\ ways$
Combination with repetition (Use combination formulas *when order doesn't matter* in the problem.)	$\boldsymbol{\frac{(n+r-1)!}{r!\,(n-1)!}}$	Where *n* is the number of things to choose from, and you choose *r* of them.	If there are 5 flavors of ice cream and you can have 3 scoops of ice cream, how many combinations can you have? You can repeat flavors. n = 5, r = 3 $\frac{(5+3-1)!}{3!\,(5-1)!} = 35\ combinations$
Combination without repetition (Use combination formulas *when order doesn't matter* in the problem.)	$\boldsymbol{\frac{n!}{r!\,(n-r)!}}$	Where *n* is the number of things to choose from, and you choose *r* of them. Sometimes you can see the following notation for the same concept: $C(n,r) = {}^nC_r = {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$	The state lottery chooses 6 different numbers between 1 and 50 to determine the winning numbers. How many combinations are possible? n = 50, r = 6 $\frac{50!}{6!\,(50-6)!} = 15{,}890{,}700\ combinations$

Examples

1) **Mr. Smith is the chair of a committee. How many ways can a committee of 4 be chosen from 9 people given that Mr. Smith must be one of the people selected?**

Mr. Smith is already chosen, so we need to choose another 3 from 8 people. In choosing a committee, order doesn't matter, so we need the combination without repetition formula.

$\frac{n!}{r!(n-r)!} = \frac{8!}{3!(8-3)!}$ = 56 ways

2) **A certain password consists of 3 different letters of the alphabet where each letter is used only once. How many different possible passwords are there?**

Order does matter in a password, and the problem specifies that you cannot repeat letters. So, you need a permutations without repetitions formula. The number of permutations of 3 letters chosen from 26 is

$\frac{n!}{(n-r)!} = \frac{26!}{(26-3)!}$ = 15,600 passwords

3) **A password consists of 3 letters of the alphabet followed by 3 digits chosen from 0 to 9. Repeats are allowed. How many different possible passwords are there?**

Order does matter in a password, and the problem specifies that you can repeat letters. So, you need a permutations with repetitions formula.

The different ways you can arrange the letters = $n^r = 26^3 =$ 17,576
The different ways you can arrange the digits = $n^r = 10^3$ = 1,000
So the number of possible passwords = 17,576 × 1,000 = 17,576,000 passwords

4) **An encyclopedia has 6 volumes. In how many ways can the 6 volumes be placed on the shelf?**

This problem doesn't require a formula from the chart. Imagine that there are 6 spots on the shelf. Place the volumes one by one.

The first volume to be placed could go in any 1 of the 6 spots. The second volume to be placed could then go in any 1 of the 5 remaining spots, and so on. So the total number of ways the 6 volumes could be placed is
6! = 6 × 5 × 4 × 3 × 2 × 1 = 720 ways

Set Theory

1. Sets

A **set** is any collection of objects. Here, the word 'object' is used in its most general sense: an object may be a diode, an aircraft, a number, or a letter, for example.
A set is often described by listing the collection of objects - these are the members or **elements** of the set. We usually write this list of elements in curly brackets, and denote the full set by a capital letter. For example,

$C = \{$the resistors produced in a factory on a particular day$\}$
$D = \{$on, off$\}$
$E = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

The elements of set C, above, are the resistors produced in a factory on a particular day. These could be individually labeled and listed individually but as the number is large it is not practical or sensible to do this. Set D lists the two possible states of a simple switch, and the elements of set E are the digits used in the decimal system.
Sometimes we can describe a set in words. For example,

'A is the set all odd numbers'.

Clearly all the elements of this set A cannot be listed.
Similarly,

'B is the set of binary digits' i.e. $B = \{0, 1\}$.

B has only two elements.

A set with a finite number of elements is called a **finite set**. B, C, D and E are finite sets. The set A has an infinite number of elements and so is not a finite set. It is called an **infinite set**.

Two sets are **equal** if they contain exactly the same elements. For example, the sets $\{9, 10, 14\}$ and $\{10, 14, 9\}$ are equal since the order in which elements are written is unimportant. Note also that repeated elements are ignored. The set $\{2, 3, 3, 3, 5, 5\}$ is equal to the set $\{2, 3, 5\}$.

Subsets

Sometimes one set is contained completely within another set. For example if $X = \{2, 3, 4, 5, 6\}$ and $Y = \{2, 3, 6\}$ then all the elements of Y are also elements of X. We say that Y is a **subset** of X and write $Y \subseteq X$.

Example 1

Given $A = \{0, 1, 2, 3\}$, $B = \{0, 1, 2, 3, 4, 5, 6\}$ and $C = \{0, 1\}$, state which sets are subsets of other sets.

Solution

A is a subset of B, that is $A \subseteq B$
C is a subset of B, that is $C \subseteq B$
C is a subset of A, that is $C \subseteq A$.

A factory produces cars over a five day period; Monday to Friday. Consider the following sets,

(a) $A = \{$cars produced from Monday to Friday$\}$

(b) $B = \{$cars produced from Monday to Thursday$\}$

(c) $C = \{$cars produced on Friday$\}$

(d) $D = \{$cars produced on Wednesday$\}$

(e) $E = \{$cars produced on Wednesday or Thursday$\}$

State which sets are subsets of other sets.

Your solution

Answer

(a) B is a subset of A, that is, $B \subseteq A$.

(b) C is a subset of A, that is, $C \subseteq A$.

(c) D is a subset of A, that is, $D \subseteq A$.

(d) E is a subset of A, that is, $E \subseteq A$.

(e) D is a subset of B, that is, $D \subseteq B$.

(f) E is a subset of B, that is, $E \subseteq B$.

(g) D is a subset of E, that is, $D \subseteq E$.

The symbol $\in$

To show that an element belongs to a particular set we use the symbol $\in$. This symbol means **is a member of** or 'belongs to'. The symbol $\notin$ means **is not a member of** or 'does not belong to'.

For example if $X = \{$all even numbers$\}$ then we may write $4 \in X$, $6 \in X$, $7 \notin X$ and $11 \notin X$.

The empty set and the universal set

Sometimes a set will contain no elements. For example, suppose we define the set K by

$$K = \{\text{all odd numbers which are divisible by 4}\}$$

Since there are no odd numbers which are divisible by 4, then K has no elements. The set with no elements is called the **empty set**, and it is denoted by $\emptyset$.

On the other hand, the set containing all the objects of interest in a particular situation is called the **universal set**, denoted by S. The precise universal set will depend upon the context. If, for example, we are concerned only with whole numbers then $S = \{\cdots - 5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$. If we are concerned only with the decimal digits then $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

The complement of a set

Given a set A and a universal set S we can define a new set, called the **complement** of A and denoted by A'. The complement of A contains all the elements of the universal set that are **not in** A.

Example 2

Given $A = \{2, 3, 7\}$, $B = \{0, 1, 2, 3, 4\}$ and $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ state
(a) A' (b) B'

Solution

(a) The elements of A' are those which belong to S but not to A.

$A' = \{0, 1, 4, 5, 6, 8, 9\}$

(b) $B' = \{5, 6, 7, 8, 9\}$

Sometimes a set is described in a mathematical way. Suppose the set Q contains all numbers which are divisible by 4 and 7. We can write

$$Q = \{x : x \text{ is divisible by 4 and } x \text{ is divisible by 7}\}$$

The symbol : stands for 'such that '. We read the above as 'Q is the set comprising all elements x, such that x is divisible by 4 and by 7'.

2. Venn diagrams

Sets are often represented pictorially by **Venn diagrams** (see Figure 1).

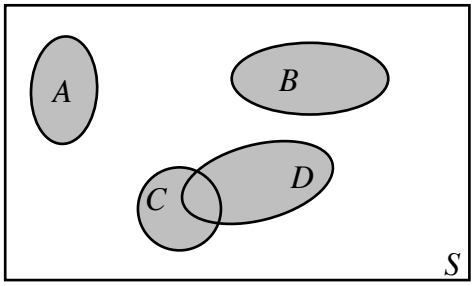


Figure 1

Here A, B, C, D represent sets. The sets A, B have no items in common so are drawn as non-intersecting regions whilst the sets C, D have some items in common so are drawn overlapping.
In a Venn diagram the universal set is represented by a rectangle and sets of interest by area regions within this rectangle.

Example 3

Represent the sets $A = \{0, 1\}$ and $B = \{0, 1, 2, 3, 4\}$ using a Venn diagram.

Solution

The elements 0 and 1 are in set A, represented by the small circle in the diagram. The large circle represents set B and so contains the elements 0,1,2,3 and 4. A suitable universal set in this case is the set of all integers. The universal set is shown by the rectangle.

Note that $A \subseteq B$. This is shown in the Venn diagram by A being completely inside B.

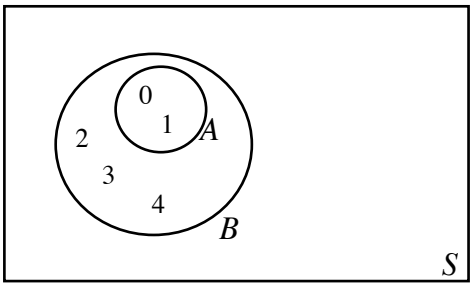


Figure 2: The set A is contained completely within B

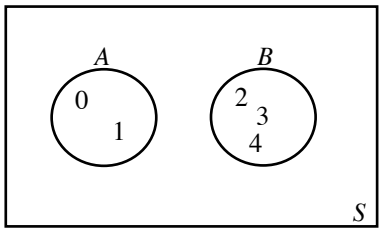
Given $A = \{0, 1\}$ and $B = \{2, 3, 4\}$ draw Venn diagrams showing
(a) A and B (b) A' (c) B'

Your solution

(a)

Answer

Note that A and B have no elements in common. This is represented pictorially in the Venn diagram by circles which are totally separate from each other as shown in the diagram.

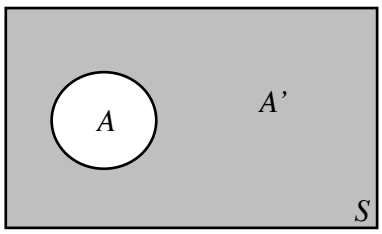


Your solution

(b)

Answer

The complement of A is the set whose elements do not belong to A. The set A' is shown shaded in the diagram.



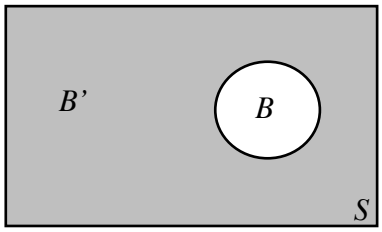
The complement of A contains elements which are not in A.

Your solution

(c)

Answer

The set B' is shown shaded in the diagram.



3. The intersection and union of sets

Intersection

Given two sets, A and B, the **intersection** of A and B is a set which contains elements that are common both to A and B. We write $A \cap B$ to denote the intersection of A and B. Mathematically we write this as:

Key Point 1

Intersection of Sets

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

This says that the intersection contains all the elements x such that x belongs to A and also x belongs to B.

Note that $A \cap B$ and $B \cap A$ are identical. The intersection of two sets can be represented by a Venn diagram as shown in Figure 3.

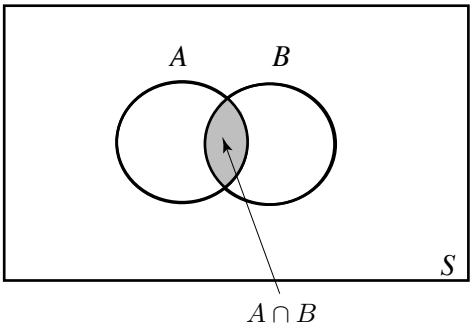


Figure 3: The overlapping area represents $A \cap B$

Example 4

Given $A = \{3, 4, 5, 6\}$, $B = \{3, 5, 9, 10, 15\}$ and $C = \{4, 6, 10\}$ state

(a) $A \cap B$, (b) $B \cap C$ and draw a Venn diagram representing these intersections.

Solution

(a) The elements common to both A and B are 3 and 5. Hence $A \cap B = \{3, 5\}$

(b) The only element common to B and C is 10. Hence $B \cap C = \{10\}$

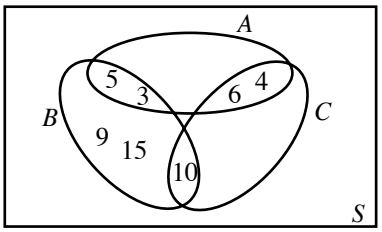


Figure 4

Given $D = \{a, b, c\}$ and $F = \{\text{the entire alphabet}\}$ state $D \cap F$.

Your solution

Answer

The elements common to D and F are a, b and c, and so $D \cap F = \{a, b, c\}$

Note that D is a subset of F and so $D \cap F = D$.

The intersection of three or more sets is possible, and is the subject of the next Example.

Example 5

Given $A = \{0, 1, 2, 3\}$, $B = \{1, 2, 3, 4, 5\}$ and $C = \{2, 3, 4, 7, 9\}$ state

(a) $A \cap B$ (b) $(A \cap B) \cap C$ (c) $B \cap C$ (d) $A \cap (B \cap C)$

Solution

(a) The elements common to A and B are 1, 2 and 3 so $A \cap B = \{1, 2, 3\}$.

(b) We need to consider the sets $(A \cap B)$ and C. $A \cap B$ is given in (a). The elements common to $(A \cap B)$ and C are 2 and 3. Hence $(A \cap B) \cap C = \{2, 3\}$.

(c) The elements common to B and C are 2, 3 and 4 so $B \cap C = \{2, 3, 4\}$.

(d) We look at the sets A and $(B \cap C)$. The common elements are 2 and 3. Hence $A \cap (B \cap C) = \{2, 3\}$.

Note from (b) and (d) that here $(A \cap B) \cap C = A \cap (B \cap C)$.

The example illustrates a general rule. For any sets A, B and C it is true that

$$(A \cap B) \cap C = A \cap (B \cap C)$$

The position of the brackets is thus unimportant. They are usually omitted and we write $A \cap B \cap C$.

Suppose that sets A and B have no elements in common. Then their intersection contains no elements and we say that A and B are **disjoint** sets. We express this as

$$A \cap B = \emptyset$$

Recall that $\emptyset$ is the empty set. Disjoint sets are represented by separate area regions in the Venn diagram.

Union

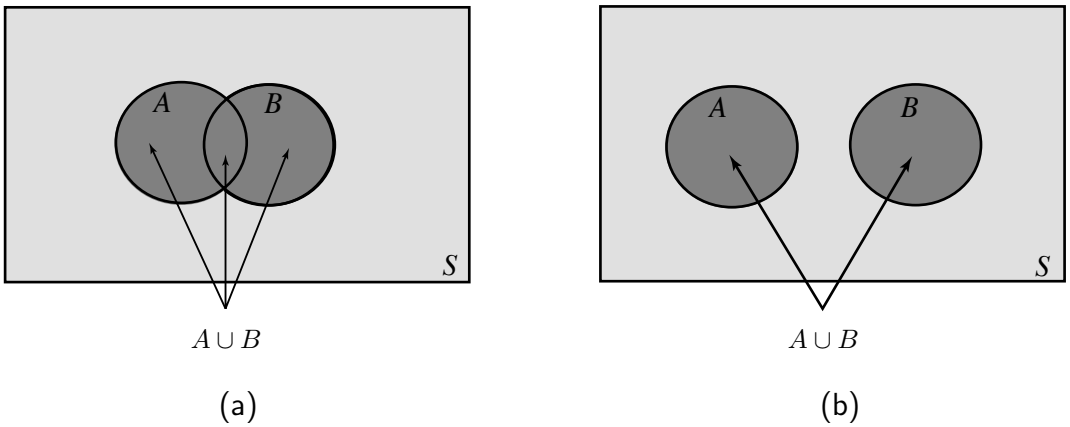
The **union** of two sets A and B is a set which contains all the elements of A together with all the elements of B. We write $A \cup B$ to denote the union of A and B. We can describe the set $A \cup B$ formally by:

Key Point 2

Union of Sets

$$A \cup B = \{x : x \in A \text{ or } x \in B \text{ or both}\}$$

Thus the elements of the set $A \cup B$ are those quantities x such that x is a member of A or a member of B or a member of both A and B. The deeply shaded areas of Figure 5 represents $A \cup B$.



(a) (b)

Figure 5

In Figure 5(a) the sets intersect, whereas in Figure 5(b) the sets have no region in common. We say they are **disjoint**.

Example 6

Given $A = \{0, 1\}$, $B = \{1, 2, 3\}$ and $C = \{2, 3, 4, 5\}$ write down

(a) $A \cup B$

(b) $A \cup C$

(c) $B \cup C$

Solution

(a) $A \cup B = \{0, 1, 2, 3\}$

(b) $A \cup C = \{0, 1, 2, 3, 4, 5\}$

(c) $B \cup C = \{1, 2, 3, 4, 5\}$.

Recall that there is no need to repeat elements in a set. Clearly the order of the union is unimportant so $A \cup B = B \cup A$.

Given $A = \{2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 8, 10\}$ and $C = \{3, 5, 7, 9, 11\}$ state
(a) $A \cup B$ (b) $(A \cup B) \cap C$ (c) $A \cap B$ (d) $(A \cap B) \cup C$ (e) $A \cup B \cup C$

Your solution

Answer

(a) $A \cup B = \{2, 3, 4, 5, 6, 8, 10\}$

(b) We need to look at the sets $(A \cup B)$ and C. The elements common to both of these sets are 3 and 5. Hence $(A \cup B) \cap C = \{3, 5\}$.

(c) $A \cap B = \{2, 4, 6\}$

(d) We consider the sets $(A \cap B)$ and C. We form the union of these two sets to obtain $(A \cap B) \cup C = \{2, 3, 4, 5, 6, 7, 9, 11\}$.

(e) The set formed by the union of all three sets will contain all the elements from all the sets:

$$A \cup B \cup C = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$$

Exercises

1. Given a set A, its complement A' and a universal set S, state which of the following expressions are true and which are false.

   (a) $A \cup A' = S$ (b) $A \cap S = \emptyset$ (c) $A \cap A' = \emptyset$
   (d) $A \cap A' = S$ (e) $A \cup \emptyset = S$ (f) $A \cup \emptyset = A$
   (g) $A \cup \emptyset = \emptyset$ (h) $A \cap \emptyset = A$ (i) $A \cap \emptyset = \emptyset$
   (j) $A \cup S = A$ (k) $A \cup S = \emptyset$ (l) $A \cup S = S$

2. Given $A = \{a, b, c, d, e, f\}$, $B = \{a, c, d, f, h\}$ and $C = \{e, f, x, y\}$ obtain the sets:

   (a) $A \cup B$ (b) $B \cap C$ (c) $A \cap (B \cup C)$
   (d) $C \cap (B \cup A)$ (e) $A \cap B \cap C$ (f) $B \cup (A \cap C)$

3. List the elements of the following sets:

   (a) $A = \{x : x$ is odd and x is greater than 0 and less than 12$\}$
   (b) $B = \{x : x$ is even and x is greater than 19 and less than 31$\}$

4. Given $A = \{5, 6, 7, 9\}$, $B = \{0, 2, 4, 6, 8\}$ and $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ list the elements of each of the following sets:

   (a) A' (b) B' (c) $A' \cup B'$
   (d) $A' \cap B'$ (e) $A \cup B$ (f) $(A \cup B)'$
   (g) $(A \cap B)'$ (h) $(A' \cap B)'$ (i) $(B' \cup A)'$

   What do you notice about your answers to (c),(g)?

   What do you notice about your answers to (d),(f)?

5. Given that A and B are intersecting sets, i.e. are not disjoint, show on a Venn diagran the following sets

   (a) A' (b) B' (c) $A \cup B'$ (d) $A' \cup B'$ (e) $A' \cap B'$

Answers

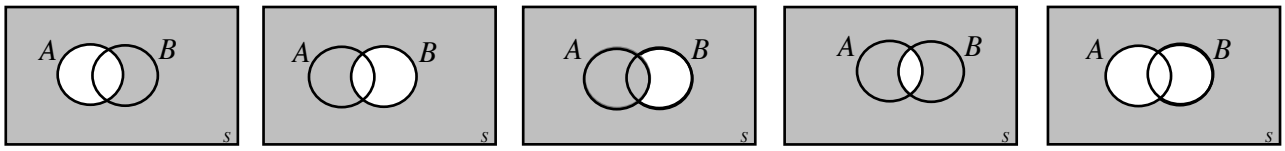
1.(a) T, (b) F, (c) T, (d) F, (e) F, (f) T, (g) F, (h) F, (i) T, (j) F, (k) F), (l) T.

2.(a) $\{a, b, c, d, e, f, h\}$, (b) $\{f\}$, (c) $\{a, c, d, e, f\}$, (d) $\{e, f\}$, (e) $\{f\}$, (f) $\{a, c, d, e, f, h\}$.

3.(a) $\{1, 3, 5, 7, 9, 11\}$, (b) $\{20, 22, 24, 26, 28, 30\}$.

4.(a) $\{0, 1, 2, 3, 4, 8\}$, (b) $\{1, 3, 5, 7, 9\}$, (c) $\{0, 1, 2, 3, 4, 5, 7, 8, 9\}$, (d) $\{1, 3\}$,
(e) $\{0, 2, 4, 5, 6, 7, 8, 9\}$, (f) $\{1, 3\}$, (g) $\{0, 1, 2, 3, 4, 5, 7, 8, 9\}$, (h) $\{1, 3, 5, 6, 7, 9\}$,
(i) $\{0, 2, 4, 8\}$.

5.



(a) (b) (c) (d) (e)

Math Formulas: Set Identities

Definitions:

Universal set : I
Empty set: $\varnothing$
Union of sets

1. $$A \cup B = \{x : x \in A \;\; or \;\; x \in B\}$$

Intersection of sets

2. $$A \cap B = \{x : x \in A \;\; and \;\; x \in B\}$$

Complement

3. $$A' = \{x \in I : x \notin A\}$$

Difference of sets

4. $$A \setminus B = \{x : x \in A \;\; and \;\; x \notin B\}$$

Cartesian product

5. $$A \times B = \{(x, y) : x \in A \;\; and \;\; y \in B\}$$

Set identities involving union

Commutativity

6. $$A \cup B = B \cup A$$

Associativity

7. $$A \cup (B \cup C) = (A \cup B) \cup C$$

Idempotency

8. $$A \cup A = A$$

Set identities involving intersection

Commutativity

9. $$A \cap B = B \cap A$$

Associativity

10. $$A \cap (B \cap C) = (A \cap B) \cap C$$

Idempotency

11. $$A \cap A = A$$

Set identities involving union and intersection

Distributivity

12. $$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

13. $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Domination

14. $$A \cap \varnothing = \varnothing$$

15. $$A \cup I = I$$

Identity

16. $$A \cup \varnothing = \varnothing$$

17. $$A \cap I = A$$

Set identities involving union, intersection and complement

Complement of intersection and union

18. $$A \cup A' = I$$

19. $$A \cap A' = \varnothing$$

De Morgan's laws

20. $$(A \cup B)' = A' \cap B\ '$$

21. $$(A \cap B)' = A' \cup B\ '$$

Set identities involving difference

22. $$B \setminus A = B \setminus (A \cup B)$$

23. $$B \setminus A = B \cap A'$$

24. $$A \setminus A = \varnothing$$

25. $$(A \setminus B) \cap C = (A \cap C) \setminus (B \cap C)$$

26. $$A' = I \setminus A$$

BINOMIAL THEOREM

Binomial Theorem for Positive Integral Index

Statement of the Theorem

Let us define ‘a’ as the first term, ‘x’ as the second term and ‘n’ as the exponent. When n is a positive integer, then

$$(a+x)^n = a^n + {}^nC_1 a^{n-1}x^1 + {}^nC_2 a^{n-2}x^2 + {}^nC_3 a^{n-3}x^3 + \ldots\ldots + {}^nC_r a^{n-r}x^r + \ldots\ldots + x^n$$

for all values of a and x.

This formula is known as Binomial Theorem.

Notes:

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

$$n! = n.(n-1).(n-2).\ldots\ldots 3.2.1$$

$$5! = 5.4.3.2.1 = 120$$

$$(a+x)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}x^1 + {}^nC_2 a^{n-2}x^2 + {}^nC_3 a^{n-3}x^3 + \ldots\ldots + {}^nC_r a^{n-r}x^r + \ldots\ldots + {}^nC_n x^n$$

Properties:

- The first term a^n and x^n last term of the expansion $(a+x)^n$.
- The total number of terms in the expansion of $(a+x)^n$ is n + 1, i.e. one more than the exponent n.
- The sum of the exponents (indices) of ‘x’ and ‘a’ in any terms is equal to n.
- The exponent of ‘a’ decreases by 1 from ‘n’ to ‘0’.
- The exponent of ‘x’ increases by 1 from ‘0’ to ‘n’.
- Since ${}^nC_0 = {}^nC_n = 1$ and ${}^nC_1 = {}^nC_{n-1} = n$ the binomial theorem can be written as:

$$(a+x)^n = a^n + na^{n-1}x^1 + {}^nC_2 a^{n-2}x^2 + {}^nC_3 a^{n-3}x^3 + \ldots\ldots + {}^nC_r a^{n-r}x^r + \ldots\ldots + nx^{n-1} + x^n$$

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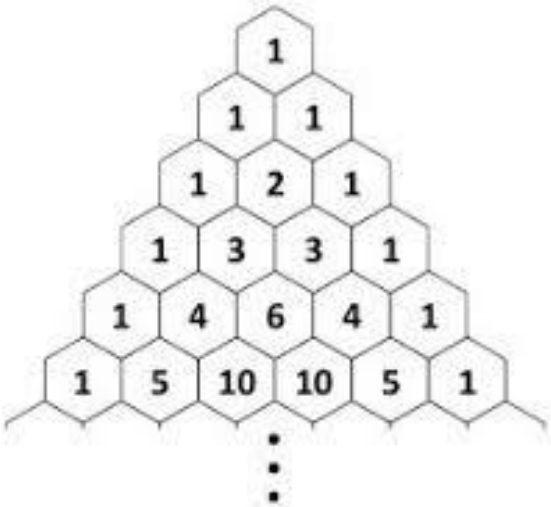
- The coefficients ${}^nC_0, {}^nC_1, {}^nC_2, \ldots\ldots {}^nC_n$ in the expansion follow a certain pattern known as Pascal's triangle.



- The coefficient of the second term and the second from the last term is n.
- The coefficients of the expansion are symmetric.
- Since ${}^nC_0, {}^nC_1, {}^nC_2, \ldots\ldots\ldots, {}^nC_n$ that occur as the coefficient of the respective terms in the expansion of $(a + x)^n$ are known as binomial coefficients in the expansion of $(a + x)^n$.
- Since ${}^nC_r = {}^nC_{r-1}$ the binomial coefficient of the terms equidistant from the beginning and the end are equal.

Example 1. Write down the expansion of the following binomial expression $(x+2y)^4$

Solution: $(x+2y)^4$

$$= x^4 + {}^4C_1.x^3.(2y)^1 + {}^4C_2.x^2.(2y)^2 + {}^4C_3.x.(2y)^3 + (2y)^4$$

$$= x^4 + \frac{4!}{1!3!}.2x^3y + \frac{4!}{2!2!}.4x^2y^2 + \frac{4!}{3!1!}.8xy^3 + 16y^4$$

$$= x^4 + \frac{4.3!}{3!}.2x^3y + \frac{4.3.2!}{2.1.2!}.4x^2y^2 + \frac{4.3!}{3!}.8xy^3 + 16y^4$$

$$= x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4$$

Notes:

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

$$n! = n.(n-1).(n-2).....3.2.1$$

$$5! = 5.4.3.2.1 = 120$$

Example 2. Write down the expansion of the following binomial expression $(3x+7)^7$

Solution:

$$(3x+7)^7$$

$$=(3x)^7+{}^7C_1.(3x)^6.(7)^1+{}^7C_2.(3x)^5.(7)^2+{}^7C_3.(3x)^4.(7)^3+{}^7C_4.(3x)^3.(7)^4+{}^7C_5.(3x)^2.(7)^5+{}^7C_6.(3x)^1.(7)^6+(7)^7$$

$$=3^7x^7+\frac{7!}{1!(7-1)!}.3^6x^6.7+\frac{7!}{2!(7-2)!}.3^5x^5.7^2+\frac{7!}{3!(7-3)!}.3^4x^4.7^3+\frac{7!}{4!(7-4)!}.3^3x^3.7^4$$

$$+\frac{7!}{5!(7-5)!}.3^2x^2.7^5+\frac{7!}{6!(7-6)!}.3^1x^1.7^6+7^7$$

$$=3^7x^7+\frac{7.6!}{6!}.3^6.x^6.7+\frac{7.6.5!}{2.5!}.3^5.x^5.7^2+\frac{7.6.5.4!}{3.2.4!}.3^4.x^4.7^3+\frac{7.6.5.4.3!}{4.3.2.3!}.3^3.x^3.7^4$$

$$+\frac{7.6.5.4.3.2!}{5.4.3.2.2!}.3^2x^2.7^5+\frac{7.6!}{6!}.3^1x^1.7^6+7^7$$

$$=2187x^7+35,721x^6+2,50,047x^5+9,72,405x^4$$

$$+22,68,945x^3+31,76,523x^2+24,70,629x+\ 8,23,543$$

APPLICATIONS OF THE BINOMIAL THEOREM

The main application of the binomial theorem is to help us to expand a binomial expression to any given power without direct multiplication more easily and conveniently. Other than that, the binomial theorem is also used in:

(a) probability theory for probabilistic analyses;

(b) higher mathematics for solving problems in algebra, calculus, combinatorics and many other areas;

(c) scientific research for solving impossible equations (e.g. Einstein equations);

(d) forecast services such as weather forecast, disaster forecast, etc;

(e) in architecture to giving shape and determining the areas of infrastructure; and

(f) giving ranks to the candidate.

SOME PARTICULAR CASES

If n is a positive integer, then

$$(a+x)^n = a^n + {}^nC_1 a^{n-1}x^1 + {}^nC_2 a^{n-2}x^2 + {}^nC_3 a^{n-3}x^3 + \ldots\ldots + {}^nC_r a^{n-r}x^r + \ldots\ldots + x^n \qquad (i)$$

In particular,

1. Replacing 'x' by (−x) in (i), we get

$$(a-x)^n = a^n - {}^nC_1 a^{n-1}x^1 + {}^nC_2 a^{n-2}x^2 - {}^nC_3 a^{n-3}x^3 + \ldots\ldots + (-1)^r\, {}^nC_r a^{n-r}x^r + \ldots\ldots + (-1)^n x^n \quad (ii)$$

2. Adding (i) and (ii), we get

$$(a+x)^n + (a-x)^n = 2[a^n + {}^nC_2 a^{n-2}x^2 + {}^nC_4 a^{n-4}x^4 + \ldots\ldots\ldots\ldots]$$

$$= 2[\text{sum of terms at odd places}]$$

3. Subtracting (ii) from (i), we get

$$(a+x)^n-(a-x)^n=2[{}^nC_1a^{n-1}x^1+{}^nC_3a^{n-3}x^3+{}^nC_5a^{n-5}x^5+.....]$$

$$=2[\text{sum of terms at even places}]$$

4. Replacing a by 1 in (i), we get

$$(1+x)^n=1+{}^nC_1x^1+{}^nC_2x^2+{}^nC_3x^3+.....+{}^nC_rx^r+.....+x^n$$

5. Replacing "a" by "1" and "x" by "(−x)" in (i), we get

$$(1-x)^n=1-{}^nC_1x^1+{}^nC_2x^2-{}^nC_3x^3+.....+(-1)^r\,{}^nC_rx^r+.....+(-1)^nx^n$$

PROPERTIES OF THE BINOMIAL COEFFICIENTS

The binomial coefficients have two important properties:

(i) The sum of all the binomial coefficients is 2^n

(ii) The sum of all the odd binomial coefficients is equal to the sum of all the even binomial coefficients and each in equal to 2^{n-1}.

Proof: (i) In the binomial expression, we have

$$(a+x)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} x^1 + {}^nC_2 a^{n-2} x^2 + {}^nC_3 a^{n-3} x^3 + \ldots\ldots + {}^nC_r a^{n-r} x^r + \ldots\ldots + {}^nC_n x^n \qquad (i)$$

where ${}^nC_0, {}^nC_1, {}^nC_2, {}^nC_3, \ldots\ldots, {}^nC_n$ are called binomial coefficients.

Putting a= x = 1 in equation no. (i), we get

$${}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n = 2^n$$

Thus the sum of all the binomial coefficients is 2^n

(ii) Putting a = 1 and x = −1 in equation no. (i), we get

$$0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + {}^nC_4 - {}^nC_5 + ...$$

$$or,\ {}^nC_0 + {}^nC_2 + {}^nC_4 + ... = {}^nC_1 + {}^nC_3 + {}^nC_5 + ...$$

Thus, the sum of all odd binomial coefficients = the sum of all even binomial coefficients.

Again, each of them $= \frac{1}{2} \times (sum \quad of \quad them)$

$$= \frac{1}{2} \times ({}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + ... + {}^nC_n)$$

$$= \frac{1}{2} \times 2^n = 2^{n-1}$$

$$i.e,\ {}^nC_0 + {}^nC_2 + {}^nC_4 + ... = {}^nC_1 + {}^nC_3 + {}^nC_5 + ...$$

General Term of $(a + x)^n$

$$(a+x)^n = {}^nC_0 a^n x^0 + {}^nC_1 a^{n-1} x^1 + {}^nC_2 a^{n-2} x^2 + {}^nC_3 a^{n-3} x^3 + \ldots\ldots + {}^nC_r a^{n-r} x^r + \ldots\ldots + {}^nC_n a^0 x^n$$

Let t_r be the rth term of the binomial expansion $(a + x)^n$ then we see that

$$t_1 = t_{0+1} = {}^nC_0 a^{n-0} x^0$$

$$t_2 = t_{1+1} = {}^nC_1 a^{n-1} x^1$$

$$t_3 = t_{2+1} = {}^nC_2 a^{n-2} x^2$$

Thus, generalizing this, we get ,

$$t_{r+1} = {}^nC_r a^{n-r} x^r$$

Notes: (a) General term of $(1 + x)^n$ is $t_{r+1} = {}^nC_r x^r$

(b) General term of $(1 - x)^n$ is $t_{r+1} = (-1)^r . {}^nC_r x^r$

Example 4. Find the general term in the expansion of $\left(x^3+\frac{1}{x^2}\right)^{12}$

Solution: General term, $t_{r+1}={}^nC_r a^{n-r}x^r$ .Here, $n=12,\quad a=x^3,\quad x=\frac{1}{x^2}$

$$t_{r+1}={}^{12}C_r.\left(x^3\right)^{12-r}.\left(\frac{1}{x^2}\right)^r$$

$$={}^{12}C_r.x^{36-3r}.\frac{1}{x^{2r}}$$

$$={}^{12}C_r.x^{36-5r}$$

Example 5. Find the general term in the expansion of $\left(x-\frac{1}{x}\right)^8$

Solution: Here, $n=8,\quad a=x,\quad x=-\frac{1}{x}$

General term,

$$t_{r+1}={}^8C_r.\left(x\right)^{8-r}.\left(-\frac{1}{x}\right)^r$$

$$=(-1)^r\,{}^8C_r.x^{8-r}.\frac{1}{x^r}$$

$$=(-1)^r\,{}^8C_r.x^{8-2r}$$

Example 6. Find 9th term of $\left(\frac{x}{2}+\frac{1}{x}\right)^{12}$

Solution: General term $t_{r+1} = {}^{n}C_r a^{n-r} x^r$

Here, $n = 12, \quad a = \frac{x}{2}, \quad x = \frac{1}{x}, \quad r = 8$

$$t_9 = t_{8+1} = {}^{12}C_8 \left(\frac{x}{2}\right)^{12-8} . \left(\frac{1}{x}\right)^8$$

$$= {}^{12}C_8 . \left(\frac{x}{2}\right)^4 . \left(\frac{1}{x}\right)^8 = \frac{12!}{8!4!}\left(\frac{x}{2}\right)^4 . \left(\frac{1}{x}\right)^8$$

$$= \frac{12\times 11\times 10\times 9}{4\times 3\times 2} . x^{4-8} . 2^{-4}$$

$$= 2^{-4} . 495 . x^{-4}$$

$$= \frac{495}{16x^4}$$

Example 7. If the number of term in $(1+x)^n$ be 11, find the 5th term and the value of n.

Solution: The total number of terms in the expression of $(1+x)^n$ is (n+ 1).

Thus n+1=11

=>n=10

The 5th term in the expression of $(1+x)^{10}$ is

$$t_{r+1} = {}^nC_r a^{n-r} x^r$$

$$n = 10, r = 4, a = 1, x = x$$

$$t_5 = t_{4+1} = {}^{10}C_4 . (1)^{10-4} . x^4$$

$$= \frac{10!}{4!(10-4)!} . x^4$$

$$= \frac{10.9.8.7.6!}{4.3.2.1.6!} x^4$$

$$= 210x^4$$

Finding Coefficient:

Example 9. Find the coefficient of x^7 in the expansion of $\left(x^2+\frac{1}{x}\right)^{11}$

Solution: Let (r+1)th term contain

$$t_{r+1} = {}^nC_r a^{n-r} x^r$$

$$n = 11, a = x^2, x = \frac{1}{x}$$

$$t_{r+1} = {}^{11}C_r.\left(x^2\right)^{11-r}.\left(\frac{1}{x}\right)^r$$

$$= {}^{11}C_r.x^{22-2r}.\frac{1}{x^r}$$

$$= {}^{11}C_r.x^{22-3r}$$

The term will contain x^7 if

22-3r = 7

$\Rightarrow$ 3r =15

$\Rightarrow$ r = 5

Therefore (5+1) th = 6th term contain x^7

$\therefore$ The coefficient $= {}^{11}C_5$

$$= \frac{11!}{6!5!}$$

$$= \frac{11.10.9.8.7.6!}{5.4.3.2.1.6!}$$

$$= 462$$

Example 10. In the expansion of $\left(5a^3 - \frac{2}{a^2}\right)^{13}$, find the coefficient of a^{-11}.

Solution: Let (r+1)th term contain a^{-11}

Now,

$$t_{r+1} = {}^nC_r a^{n-r} x^r$$

$$n = 13, a = 5a^3, x = \left(-\frac{2}{a^2}\right)$$

$$t_{r+1} = {}^{13}C_r.\left(5a^3\right)^{13-r}.\left(-\frac{2}{a^2}\right)^r$$

$$= {}^{13}C_r.5^{13-r}.a^{39-3r}.\frac{(-2)^r}{a^{2r}}$$

$$= 5^{13-r}.(-2)^r.{}^{13}C_r.a^{39-5r}$$

The term will contain a^{-11} if 39-5r = -11

=> 5r =50

=> r = 10

Therefore (10+1) th = 11th term contain a^{-11}

$\therefore$ The coefficient $= 5^{13-10}.(-2)^{10}.{}^{13}C_{10}$

$$= 5^3.2^{10}.\frac{13!}{10!3!}$$

$$= 5^3.2^{10}.\frac{13.12.11.10!}{10!.3.2.1}$$

$$= 36608000$$

Finding Independent Term

Example 12. Find the term independent of x in the expansion of $\left(x^2 - \frac{2}{x}\right)^6$

Solution: Let (r + 1)th term is independent of x in the expansion of $\left(x^2 - \frac{2}{x}\right)^6$

Now $t_{r+1} = {}^nC_r a^{n-r} x^r$

$$n = 6, a = x^2, x = -\frac{2}{x}$$

$$t_{r+1} = {}^6C_r . \left(x^2\right)^{6-r} . \left(-\frac{2}{x}\right)^r$$

$$= (-2)^r . {}^6C_r . x^{12-2r} . \frac{1}{x^r}$$

$$= (-2)^r . {}^6C_r . x^{12-3r}$$

Clearly, the (r + 1)th term will be independent of x if 12 − 3r = 0

or 3r = 12

or r = 4

Therefore, (4 + 1)th = 5th term is independent of 'x' and its value

$$= (-2)^4 . {}^6C_4 = 16 . \frac{6!}{4!2!}$$

$$= 16 . \frac{6.5.4!}{2.1.4!} = 240$$

Example 13. Find the term independent of x in $\left(2x-\frac{1}{\sqrt{x}}\right)^{15}$

Solution: Let (r+1)th term be t_{r+1} independent of x,

Now,

$$t_{r+1} = {}^{n}C_r a^{n-r} x^r$$

$$n = 15, a = 2x, x = -\frac{1}{\sqrt{x}}$$

$$t_{r+1} = {}^{15}C_r.(2x)^{15-r}.\left(-\frac{1}{\sqrt{x}}\right)^r$$

$$= (-1)^r.(2)^{15-r}.{}^{15}C_r.x^{15-r}.\frac{1}{x^{\frac{r}{2}}}$$

$$= (-1)^r.(2)^{15-r}.{}^{15}C_r.x^{15-\frac{3r}{2}}$$

The (r + 1)th term will be independent of x if

$$15-\frac{3r}{2}=0$$

$$\Rightarrow \frac{3r}{2}=15 \Rightarrow r=10$$

Therefore, (10 + 1)th = 11th term is independent of 'x' and its value

$$= (-1)^{10}(2)^{15-10}.{}^{15}C_{10}$$

$$= (2)^5.\frac{15!}{10!5!}$$

$$= 32.\frac{15.14.13.12.11.10!}{10.!5.4.3.2.1}$$

$$= 96096$$

MIDDLE TERM (GREATEST CO-EFFICIENT)

The middle term of any expansion depends upon the value of index ‘n’.

(i) If the value of index ‘n’ is even:

The total number of terms in the expansion of $(a+x)^n$ is (n+1) which is odd. So, there is only one middle term and it is $\left(\frac{(n+1)+1}{2}\right)th$ term, i.e. $\left(\frac{n}{2}+1\right)th$ term.

Therefore, the required middle term

$$t_{r+1} = {}^nC_r a^{n-r} x^r$$

$$t_{\left(\frac{n}{2}+1\right)} = {}^nC_{\frac{n}{2}} . (a)^{\frac{n}{2}} . (x)^{\frac{n}{2}}$$

Example 15: Find the middle term in the expansion of $(a+x)^8$

Solution: Here, the value of index 'n' = 8, which is even.

So, the expansion has one middle term and it is $\left(\frac{8}{2}+1\right)th$ term, i.e. 5th term.
Therefore, the required middle term

$$t_{\left(\frac{n}{2}+1\right)} = {}^nC_{\frac{n}{2}}.(a)^{\frac{n}{2}}.(x)^{\frac{n}{2}}$$

$$t_{r+1} = {}^nC_r a^{n-r} x^r$$

$$t_5 = t_{4+1} = {}^8C_4.a^4.x^4$$

$$= \frac{8!}{4!(8-4)!}a^4.x^4$$

$$= \frac{8.7.6.5.4!}{4.3.2.4!}a^4.x^4$$

$$= 70a^4.x^4$$

Example 16: Find the middle term in the expansion of $\left(2px-\frac{q}{x^2}\right)^{12}$

Solution: Here, the value of index = 12, which is even. So, the expansion has one middle term and it is term, $\left(\frac{12}{2}+1\right)th$ i.e. 7th term

$$t_{r+1} = {}^{n}C_r a^{n-r} x^r$$

$$t_7 = t_{6+1} = {}^{12}C_6.(2px)^{12-6}\left(-\frac{q}{x^2}\right)^6$$

$$= \frac{12!}{6!6!}(2px)^6\left(\frac{q}{x^2}\right)^6$$

$$= \frac{12.11.10.9.8.7.6!}{6.5.4.3.2.1.6!}\times 64\times p^6x^6\times\frac{q^6}{x^{12}}$$

$$= 59136\frac{p^6q^6}{x^6}$$

(ii) If the value of index ‘n’ is odd:

The total number of terms in the expansion of $(a+x)^n$ is (n+1) which is even. So, there are two middle term and they are $\left(\frac{(n+1)}{2}\right)th$ term and $\left(\frac{(n+1)}{2}+1\right)th$, i.e. $\left(\frac{n+3}{2}\right)th$ Term.

Therefore, the required first middle term

$$t_{r+1} = {}^nC_r a^{n-r} x^r$$

$$t_{\left(\frac{n+1}{2}\right)} = t_{\left(\frac{n-1}{2}+1\right)} = {}^nC_{\frac{n-1}{2}}.(a)^{\frac{n+1}{2}}.(x)^{\frac{n-1}{2}}$$

Second middle term

$$t_{r+1} = {}^nC_r a^{n-r} x^r$$

$$t_{\left(\frac{n+3}{2}\right)} = t_{\left(\frac{n+1}{2}+1\right)} = {}^nC_{\frac{n+1}{2}}.(a)^{\frac{n-1}{2}}.(x)^{\frac{n+1}{2}}$$

Example 17: Find the middle term in the expansion of $(a+x)^9$

Solution: Here, the value of index 'n' = 9, which is odd.

So, the expansion have two middle term and it is $\left(\frac{9+1}{2}\right)th$term, i.e. 5th term and $\left(\frac{9+3}{2}\right)th$ i.e. 6th term.
Therefore, the required first middle term

$$t_{r+1} = {}^nC_r a^{n-r} x^r$$

$$t_5 = t_{4+1} = {}^9C_4 . a^5 . x^4 = \frac{9!}{4!(9-4)!} a^5 . x^4$$

$$= \frac{9.8.7.6.5!}{4.3.2.5!} a^5 . x^4 = 186 a^5 . x^4$$

Therefore, the required second middle term

$$t_6 = t_{5+1} = {}^9C_5 . a^4 . x^5 = \frac{9!}{5!(9-5)!} a^4 . x^5$$

$$= \frac{9.8.7.6.5!}{4.3.2.5!} a^4 . x^5 = 186 a^4 . x^5$$

Example 18: Find the middle term in the expansion of $\left(x-\frac{1}{x}\right)^{13}$

Solution: Here, the value of index 'n' = 13, which is odd.

So, the expansion have two middle term and it is $\left(\frac{13+1}{2}\right)th$ term, i.e. 7th term and $\left(\frac{13+3}{2}\right)th$ i.e. 8th term.

Therefore, the required first middle term

$$t_{r+1} = {}^{n}C_r a^{n-r} x^r$$

$$t_7 = t_{6+1} = {}^{13}C_6 . x^{13-6} . \left(-\frac{1}{x}\right)^6 = \frac{13!}{6!(13-6)!} x^7 . \frac{1}{x^6}$$

$$= \frac{13.12.11.10.9.8.7!}{6.5.4.3.2.1.7!} x^{7-6} = 1716x$$

Therefore, the required second middle term

$$t_8 = t_{7+1} = {}^{13}C_7 . x^{13-7} . \left(-\frac{1}{x}\right)^7 = -\frac{13!}{7!(13-7)!} x^6 . \frac{1}{x^7}$$

$$= -\frac{13.12.11.10.9.8.7!}{6.5.4.3.2.1.7!} x^{6-7} = -1716x^{-1}$$

13. If the fifth term of the expansion $\left(a^{2/3}+\frac{1}{a}\right)^n$ does not contain 'a', then n is equal to:

(a) 10 (b) 12 (c) 9 (d) 11.

Solution: $t_{r+1} = {}^nC_r a^{n-r} x^r$

$$t_5 = t_{4+1} = {}^nC_4 \left(a^{2/3}\right)^{n-4} \left(\frac{1}{a}\right)^4$$

$$= {}^nC_4 \times a^{\frac{2n-8}{3}} \times \frac{1}{a^4}$$

$$= {}^nC_4 \times a^{\frac{2n-20}{3}}$$

Now,

$$\frac{2n-20}{3} = 0$$

$$\Rightarrow 2n-20=0$$

$$\Rightarrow n=10$$

14. Find the value of $^{7}C_1 + {}^{7}C_2 + {}^{7}C_3 + ... + {}^{7}C_7$

(a) 128 (b) 127 (c) 126 (d) 129

Solution: $^{7}C_1 + {}^{7}C_2 + {}^{7}C_3 + ... + {}^{7}C_7$

$= 1 + {}^{7}C_1 + {}^{7}C_2 + {}^{7}C_3 + ... + {}^{7}C_7 - 1$

$= {}^{7}C_0 + {}^{7}C_1 + {}^{7}C_2 + {}^{7}C_3 + ... + {}^{7}C_7 - 1$

$= 2^7 - 1 = 128 - 1 = 127$

15. Find the value of $^{15}C_1 - {}^{15}C_2 + {}^{15}C_3 - {}^{15}C_4 ..... + (-1)^{15}\, {}^{15}C_{15}$

(a) 10 (b) 12 (c) 0 (d) 11.

Solution: $^{15}C_1 - {}^{15}C_2 + {}^{15}C_3 - {}^{15}C_4 ..... + (-1)^{15}\, {}^{15}C_{15}$

$= {}^{15}C_1 + {}^{15}C_3 + ... + (-1)^{15}\, {}^{15}C_{15} - ({}^{15}C_2 + {}^{15}C_4 + ... + {}^{15}C_{14})$

$= 2^{15-1} - 2^{15-1}$

$= 0$

16. Find the coefficient of x in the expansion of $\left(1-2x^3+3x^5\right)\left(1+\frac{1}{x}\right)^8$

(a) 10 (b) 12 (c) 0 (d) 11.

Solution: $\left(1-2x^3+3x^5\right)\left(1+\frac{1}{x}\right)^8$

$$=\left(1-2x^3+3x^5\right)\left(1+{}^8C_1\frac{1}{x}+{}^8C_2\frac{1}{x^2}+{}^8C_3\frac{1}{x^3}+{}^8C_4\frac{1}{x^4}+{}^8C_5\frac{1}{x^5}+{}^8C_6\frac{1}{x^6}+{}^8C_7\frac{1}{x^7}+\frac{1}{x^8}\right)$$

$\therefore$ Coefficient of $x=-2\times{}^8C_2+3\times{}^8C_4$ and x has no other coefficient in the product.

$\therefore$ Required coefficient = –56 + 210 = 154

LOGARITHM

Definition of Logarithms

- Consider the equation $a^x = N$
  where $a > 0$, $a \neq 1$

- Then x is said to be the logarithm of N to the base a.

- written as $x = \log_a N$

- read as x is logarithm of N to the base a

- a is called the base
- x is called the index

Example:

$$2^4 = 16$$

$$4 = \log_2 16$$

$$2^{-3} = \frac{1}{8}$$

$$-3 = \log_2 \frac{1}{8}$$

Special cases:

(i) $\boxed{a^0 = 1 \quad \text{then} \quad 0 = \log_a 1}$

Logarithm of 1 to any non zero base is zero.

example : $\boxed{\log_5 1 = 0}$

(ii) $\boxed{a^1 = a \quad \text{then} \quad 1 = \log_a a}$

Logarithm to any number to itself as base is 1

example :

$$\boxed{\log_4 4 = 1}$$

Laws of Logarithm

Law 1 : $$\log_a(m \times n) = (\log_a m) + (\log_a n)$$

Law 2 : $$\log_a\left(\frac{m}{n}\right) = (\log_a m) - (\log_a n)$$

Law 3 : $$\log_a(m^n) = n \times (\log_a m)$$

Change of Base

$$\log_a m = \log_b m \times \log_a b$$

Corollary :

$$\log_b a \times \log_a b = 1$$

$$\frac{\log_b m}{\log_b a} = \log_a m$$

Example 1

Find the logarithm of 2025 to the base $3\sqrt{5}$.

$$
\begin{aligned}
2025 &= 3^4 \times 5^2 \\
&= 3^4 \times (\sqrt{5})^4 \\
&= (3\sqrt{5})^4
\end{aligned}
$$

$$
\begin{aligned}
&\log_{3\sqrt{5}} 2025 \\
&= \log_{3\sqrt{5}} (3\sqrt{5})^4 \\
&= 4 \log_{3\sqrt{5}} 3\sqrt{5} \\
&= 4
\end{aligned}
$$

Example 2

The logarithm of a number to the base $\sqrt{2}$ is k.
What is its logarithm to the base $2\sqrt{2}$.

$$\log_{\sqrt{2}} x = k$$

$$\Rightarrow \quad \log_{2\sqrt{2}} x \times \log_{\sqrt{2}} 2\sqrt{2} = k$$

$$\Rightarrow \quad \log_{2\sqrt{2}} x = \frac{k}{\log_{\sqrt{2}} 2\sqrt{2}}$$

$$= \frac{k}{\log_{\sqrt{2}} (\sqrt{2})^3}$$

$$= \frac{k}{3}$$

Properties of logarithms

Let *m* and *n* be arbitrary positive numbers such that a > 0, a ≠ 1, b > 0, b ≠ 1 then

(1) $\log_a a = 1,\ \log_a 1 = 0$

(2) $\log_a b . \log_b a = 1 \Rightarrow \log_a b = \dfrac{1}{\log_b a}$

(3) $\log_c a = \log_b a.\ \log_c b$ or $\log_c a = \dfrac{\log_b a}{\log_b c}$

(4) $\log_a(mn) = \log_a m + \log_a n$

(5) $\log_a\left(\dfrac{m}{n}\right) = \log_a m - \log_a n$

(6) $\log_a m^n = n\log_a m$

(7) $a^{\log_a m} = m$

(8) $\log_a\left(\dfrac{1}{n}\right) = -\log_a n$

(9) $\log_{a^\beta} n = \dfrac{1}{\beta}\log_a n$

(10) $\log_{a^\beta} n^\alpha = \dfrac{\alpha}{\beta}\log_a n$, $(\beta \neq 0)$

(11) $a^{\log_c b} = b^{\log_c a}$, $(a, b, c > 0 \text{ and } c \neq 1)$

COMPOUND INTEREST AND ANNUITIES

How it works.

Simple interest is interest on the principle amount.

Compound interest is when your principle *and* any earned interest both earn interest.

Calculate compound interest using this formula:

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

A—Total amount

P —principle amount

r —interest rate

m —number of compounding periods

t —time in years

Consider this example: You begin with ₹100 invested at 10% annual interest.

After	Simple Interest	Compound Interest
1 year	110	110
2 years	120	121
3 years	130	133
4 years	140	146
5 years	150	161
10 years	200	259
20 years	300	672
50 years	**600**	**11,739**

Example: ₹100 is invested at 10% interest compounded yearly for 6 years

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

$$P = 100, \quad r = \frac{10}{100} = 0.1$$
$$m = 1, \quad t = 6$$

$$A = 100(1 + 0.1)^6$$
$$= 177.16$$

₹ 250 invested at 6.5% for 8 years compounded monthly.

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

$$P = 250, \quad r = \frac{6.5}{100} = 0.065$$
$$m = 12, \quad t = 8$$

$$\begin{aligned} A &= 250\left(1 + \frac{0.065}{12}\right)^{12\times 8} \\ &= 250(1.0054167)^{96} \\ &= 419.9174 \end{aligned}$$

Try these:

1. $750 at 6.5% for 5 years compounded annually
2. $25,000 at 8% for 3 years compounded annually
3. $680 at 5.5% for 1.5 years compounded monthly
4. $1500 at 4.5% for 2 years compounded monthly

Depreciation :

in case of depreciation the principal value is diminished every period by a certain amount.

$$A = P(1 - i)^{mt}$$

Example : A machine depreciates 10% p.a. for first two years and then 7% p.a. for the next three years, depreciation being calculated on the diminishing value. If the value of the machine be Rs. 10,000 initially, find the average rate of depreciation and the depreciated value of the machine at the end of the fifth year.

Example : A machine depreciates at the rate of 10% of its value at beginning of a year. The machine was purchased for Rs. 5,810 and the scrap value realized when sold was Rs. 2,250. Find the number of years the machine was used.

$$P = 5810, A = 2250, i = 0.1$$

$$n = ?$$

$$A = P(1-i)^n$$

$$\Rightarrow 2250 = 5810(1-0.1)^n$$

$$\Rightarrow \log(2250) = \log 5810 + n\log(0.9)$$

$$\Rightarrow n = \frac{\log 2250 - \log 5810}{\log 0.9}$$

$$= 9 \quad (\text{ approx. })$$

Effective rate : $$\left\{\left(1+\frac{r}{m}\right)^{m}-1\right\}\times 100$$

Problem: What is the present value of Rs. 1000 due in two years at 5% Compound interest, according as the interest is paid

(a) Yearly

(b) Half-yearly

Problem: A sum of money invested at C.I. payable yearly amounts to Rs. 10,816 at the end of the second year and to Rs. 11,248.64 at the end of the third year. Find the rate of interest and the sum.

Problem: A man left Rs. 18,000 with the direction that it should be divided in such a way that his three sons aged 9, 12 and 15 years should receive the same amount when they reach the age of 25. If the rate of interest is $3\frac{3}{2}\%$ p.a., what should each son receive when he is 25 years old?

Problem: A man can buy a flat for Rs. 1,00,000 cash, or for Rs. 50,000 down and Rs. 60,000 at the end of one year. If money is worth 10% per year compounded half yearly, which plan should he choose?

Problem: Purchasing of National Saving Certificate makes the investment double itself in 6 years. Find the rate of interest accrued; if compounded half yearly.

Problem: In a certain population, annual birth and death rates per 1000 are 29.4 and 9.4 respectively. Find the number of years in which population will be triple assuming that there is no immigration or emigration.

CORRELATION and ASSOCIATION

In this chapter we shall be primarily interested in the following:

- Whether the change in one variable is related with the change in the other variable, on the average.
- Whether the change in one attribute is associate with the change in the other attribute.

The first type of problem mentioned in
(a) is known as the problem of correlation and
the second type of problem mentioned in
(b) is known as the problem of Association.

In Correlation, we determine the nature and degree of relationship between two variables whereas in Association we determine the nature and degree of relationship between two attribute.

BIVARIATE DATA

- Bivariate data refer to data relating to two variables. Statistical data relating to simultaneous measurement of two variables are called bivariate data. The simultaneous measurement of two variables of x and y, i.e. (x, y). Thus, for n items we have n pairs of measurements or observations as $(x_1, y_1), (x_2, y_2), \ldots\ldots, (x_n, y_n)$. On the basis of the above analysis, bivariate data can be defined as follows:
- Example:

Bivariate Data Showing Height and Weight of 15 Individuals

Height (inches)	*Weight (lbs.)*	*Height (Inches)*	*Weight (lbs.)*
68	135	66	131
64	127	70	149
69	149	70	130
62	133	59	118
67	145	72	139
63	114	71	136
67	152	67	136
68	157		

The above data show that height (in inches) and weight (in lbs.) for the 1st individual are (68, 135), for the 2nd individual (64, 127), ..., for the last individual (67, 136).

Table 3.17 shows the frequency distribution of two variables, namely, age and marks obtained by 50 students in an intelligent test. Classes defined for marks are arranged horizontally (rows) and the classes defined for age are arranged vertically (columns). each cell shows the frequency of the corresponding row and column values. for instance, there are 5 students whose age fall in the class 20 – 22 years and their marks lie in the group 30 – 40.

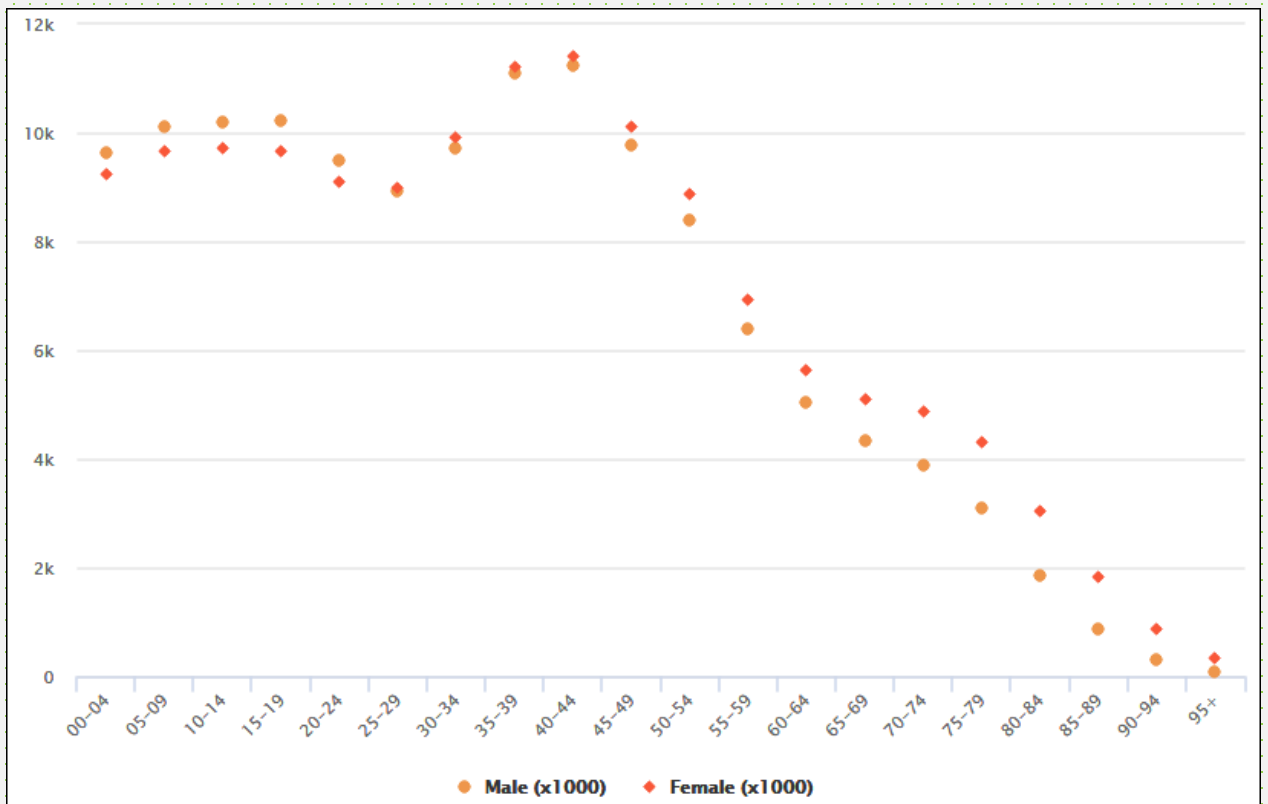
Table 3.17

Bivariate Frequency Distribution of Age and Marks

Marks	Age in Years				Total
	16 – 18	18 - 20	20 - 22	22 – 24	
10 – 20	2	1	1	-	4
20 – 30	3	2	3	1	9
30 – 40	3	3	5	6	17
40 – 50	2	2	3	4	11
50 – 60	-	1	2	2	5
60 – 70	-	1	2	1	4
Total	**10**	**10**	**16**	**14**	**50**

SCATTER DIAGRAM

- A **scatter diagram** (Also known as **scatter plot**, **scatter graph**, and correlation chart) is a tool for analyzing relationships between two variables for determining how closely the two variables are related. One variable is plotted on the horizontal axis and the other is plotted on the vertical axis. The pattern of their intersecting points can graphically show relationship patterns.

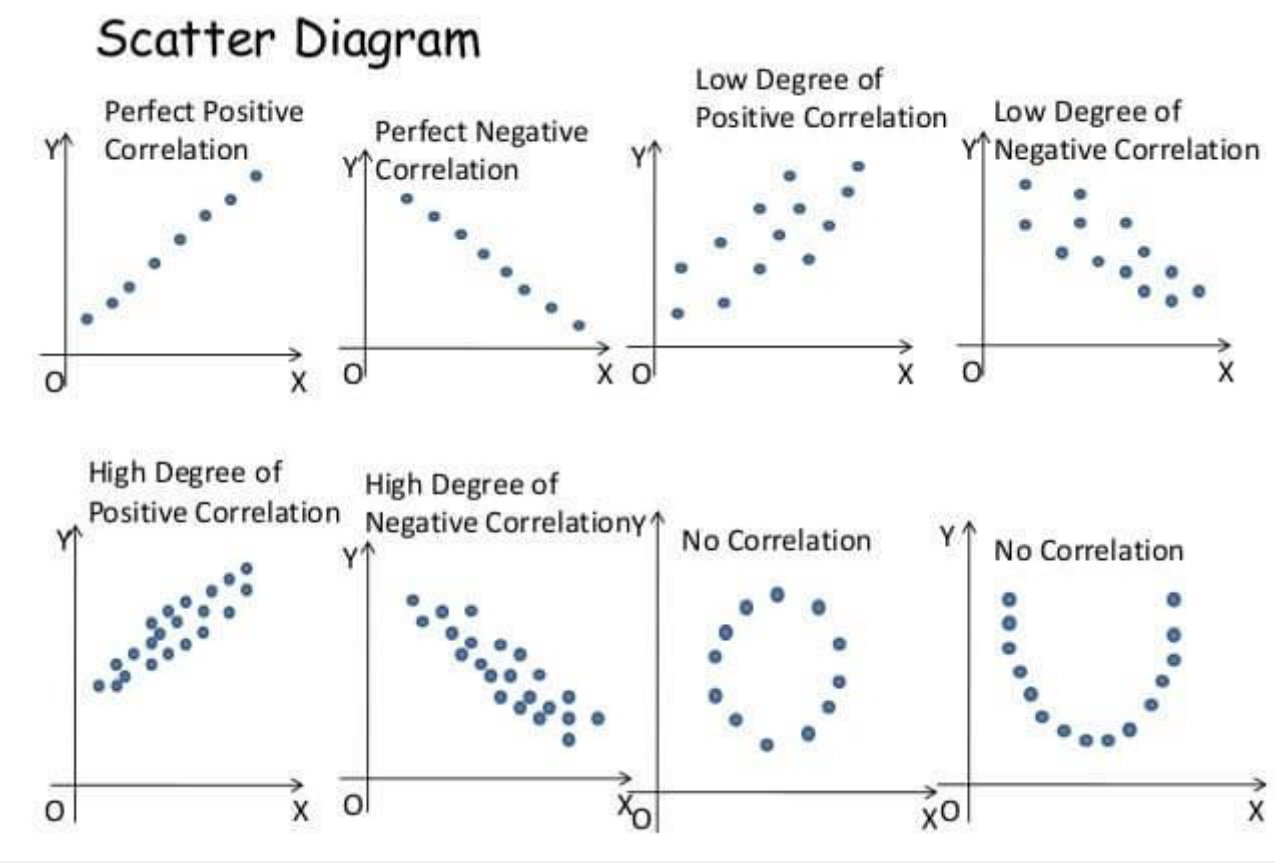


United States Population of Year 2000

SCATTER DIAGRAM CORRELATION PATTERNS

The degree to which the variables are related to each other depends on how the points are scattered over the chart. The more the points plotted are scattered over the chart, the lesser is the degree of correlation between the variables. The more the points plotted are closer to the line, the higher is the degree of correlation. The degree of correlation is denoted by “r”.
The following types of scatter diagrams show in the table tell about the degree of correlation between variable X and variable Y.

Correlation Pattern	X / Y Values
Perfect Positive Correlation	The value of Y increases as the value of X increases.
Perfect Negative Correlation	The value of Y decreases as the value of X increases.
Low Degree of Positive Correlation	The value of Y increases slightly as the value of X increases.
Low Degree of Negative Correlation	The value of Y decreases slightly as the value of X increases.
Complex Correlation	The value of Y seems to be related to the value of X, but the relationship is not easily determined.
No Correlation	There is no demonstrated connection between the two variables.



COVARIANCE

- For a set of n pairs of observations $(x_1, y_1), (x_2, y_2), \ldots\ldots, (x_n, y_n)$, the covariance of x and y, denoted by cov(x, y), is defined by

$$\operatorname{cov}(x, y) = \frac{1}{n}\sum(x-\bar{x})(y-\bar{y})$$

On simplification, we have

$$\operatorname{cov}(x, y) = \frac{1}{n}\sum(x-\bar{x})(y-\bar{y})$$

$$= \frac{1}{n}\sum(xy - \bar{x}y - x\bar{y} + \bar{x}\,\bar{y})$$

$$= \frac{\sum xy}{n} - \bar{x}\frac{\sum y}{n} - \bar{y}\frac{\sum x}{n} + \frac{\bar{x}\,\bar{y}n}{n}$$

$$= \frac{\sum xy}{n} - \bar{x}\,\bar{y} - \bar{x}\,\bar{y} + \bar{x}\,\bar{y}$$

$$\operatorname{cov}(x, y) = \frac{\sum xy}{n} - \bar{x}\,\bar{y},$$ For independent variables x and y cov(x,y)=0

MEASURES OF CORRELATION

The degree of correlation between two variables x and y is measured by correlation coefficient. It is a relative measure and it has no unit and it determine by two methods:

- Product-moment or covariance method
- Rank method.

Product-moment correlation coefficient is denoted by r and rank correlation is denoted by R. r and R has no unit i. e. they are independent of the units of the variables x and y.

PRODUCT-MOMENT CORRELATION COEFFICIENT

The measure of product-moment correlation-coefficient between two variables x and y, according to Karl Pearson, is denoted by r and is defined as

$$r = \frac{\text{cov}(x, y)}{\sigma_x \sigma_y}$$

Where σ_x *and* σ_y are the standard deviation of the variables x and y respectively and cov (x, y) is the covariance between x and y.

Example: If S.D. of x=5 and S.D. of y= 2 and cov (x, y)=8, then the correlation coefficient between x and y is

$$r = \frac{\text{cov}(x, y)}{\sigma_x \sigma_y} = \frac{8}{5 \times 2} = \frac{8}{10} = 0.8$$

A. When deviations are taken from actual mean

Karl Pearson's correlation coefficient formula can be written in various other forms.

$$r = \frac{\operatorname{cov}(x, y)}{\sigma_x \sigma_y}$$

$$\Rightarrow r = \frac{\frac{1}{n}\sum(x-\bar{x})(y-\bar{y})}{\sqrt{\frac{1}{n}\sum(x-\bar{x})^2}\sqrt{\frac{1}{n}\sum(y-\bar{y})^2}}$$

$$\Rightarrow r = \frac{\frac{\sum xy}{n} - \left(\frac{\sum x}{n}\right)\left(\frac{\sum y}{n}\right)}{\sqrt{\frac{1}{n}\sum\left(x^2 - 2x.\bar{x} + \bar{x}^2\right)}\sqrt{\frac{1}{n}\sum\left(y^2 - 2y.\bar{y} + \bar{y}^2\right)}}$$

$$\Rightarrow r = \frac{\sum xy - n\overline{xy}}{\sqrt{\sum x^2 - n\bar{x}^2}\sqrt{\sum y^2 - n\bar{y}^2}}$$

SOME IMPORTANT THEOREM

- Theorem 1: The correlation coefficient is independent of changes of origin and scale of the observations.
- Theorem 2: The correlation coefficient r lies between -1 and +1, i.e., $-1 \le r \le 1$
- Theorem 3: If the variable x and y are connected by the linear relation y=a+ bx, then $r_{xy} = 1 or -1$ according to b is positive or negative.
- Theorem 4: For any two variables x and y $\operatorname{var}(ax+by) = a^2 \operatorname{var}(x) + b^2 \operatorname{var}(y) + 2ab \operatorname{cov}(x,y)$ where a and b are two constants.

PROPERTIES OF CORRELATION COEFFICIENT

1. The correlation coefficient is independent of change of origin and change of scale of observation.

2. The correlation coefficient is a pure number, i.e., it has no unit.

3. The correlation coefficient r lies between -1 and +1, $-1 \leq r \leq 1$

4. If two variables are independent of each other, then the correlation between them is zero, i.e., r=0

5. The value of coefficient of correlation ‘r’ remains unchanged when a constant is added or subtracted on or from one or both variables.

Example No.1: Calculate Karl Pearson's Coefficient from the following information. (April-2018)

Supply(tons)	7	9	13	16	19	21	25	34
Price(thousand)	11	13	16	16	19	26	23	36

Solution: Let X be the Supply in tons and Y be the Price in thousand.

We use direct method and prepare the following table.

X	Y	X^2	Y^2	XY
7	11	49	121	77
9	13	81	169	117
13	16	169	256	208
16	16	256	256	256
19	19	361	361	361
21	26	441	676	546
25	23	625	529	575
34	36	1156	1296	1224
$\sum y =$ **144**	$\sum y =$ **160**	$\sum x^2 =$ **3138**	$\sum y^2 =$ **3664**	$\sum xy =$ **3364**

Here n = 8, Hence $\bar{x} = \frac{\sum x}{n} = \frac{144}{8} = 18$ and $\bar{y} = \frac{\sum y}{n} = \frac{160}{8} = 20$.

$$r = \frac{\sum xy - n\bar{x}\bar{y}}{\sqrt{(\sum x^2 - n\bar{x}^2)(\sum y^2 - n\bar{y}^2)}}$$

$$r = \frac{3364 - 8 \times 18 \times 20}{\sqrt{(3138 - 8 \times 324)(3664 - 8 \times 400)}}$$

$$r = \frac{3364 - 2880}{\sqrt{546 \times 464}}$$

$$r = \frac{484}{503.3328}$$

$$r = 0.9616$$

Interpretation- There is a strong uphill (positive) correlation between supply and price.

Example No.2: Following are the values of import of raw material and export of finished products in suitable units. (April-2011, 2013)

Export	10	11	14	14	20	22	16	12	15	13
Import	12	14	15	16	21	26	21	15	16	14

Solution: Let X be the Exported quantity and Y be the imported quantity.

We use direct method and prepare the following table.

X	Y	X^2	Y^2	XY
10	12	100	144	120
11	14	121	196	154
14	15	196	225	210
14	16	196	256	224
20	21	400	441	420
22	26	484	676	572
16	21	256	441	336
12	15	144	225	180
15	16	225	256	240
13	14	169	196	182
$\sum x =$ **147**	$\sum y =$ **170**	$\sum x^2 =$ **2291**	$\sum y^2 =$ **3056**	$\sum xy =$ **2638**

Here n = 10, Hence $\bar{x} = \frac{\sum x}{n} = \frac{147}{10} = 14.7$ and $\bar{y} = \frac{\sum y}{n} = \frac{170}{10} = 17$

$$r = \frac{\sum xy - n\bar{x}\bar{y}}{\sqrt{(\sum x^2 - n\bar{x}^2)(\sum y^2 - n\bar{y}^2)}}$$

$$r = \frac{2638 - 10 \times 14.7 \times 17}{\sqrt{(2291 - 10 \times 216.09)(3056 - 10 \times 289)}}$$

$$r = \frac{2638 - 2499}{\sqrt{130.1 \times 166}}$$

$$r = \frac{139}{146.9578}$$

$$r = 0.9458$$

Interpretation- There is a strong uphill (positive) correlation between import of raw material and export of finished products.

B. When deviations are taken from zero, i.e. calculation of *r* from original/raw scores (Direct Method)

It is an another method with ungrouped data, which does not require the use of deviations. It deals entirely with original scores when values of x and y are small. The formula may look forbidding but is really easy to apply.

The formula in such a case is:

$$r_{xy} = \frac{n \cdot \Sigma xy - (\Sigma x)(\Sigma y)}{\sqrt{\left[n \cdot \Sigma x^2 - (\Sigma x)^2\right]\left[n \cdot \Sigma y^2 - \Sigma y^2\right]}} \tag{6.3}$$

This formula is suitable in the following cases:

(*i*) When mean values are in fractions
(*ii*) To compute *r* from direct raw scores
(*iii*) When data are small ungrouped
(*iv*) When a good calculating machine is available

Calculation of r from original score (Direct method)

Find the coefficient of correlation of the following data:

x :	9	5	8	6	7	4	5	6
y :	7	4	6	8	9	5	2	4

Solution: **Computation of coefficient of correlation**

x	y	x^2	y^2	xy
9	7	81	49	63
5	4	25	16	20
8	6	64	36	48
6	8	36	64	48
7	9	49	81	63
4	5	16	25	20
5	2	25	4	10
6	4	36	16	24
$\Sigma x = 50$	$\Sigma y = 45$	$\Sigma x^2 = 332$	$\Sigma y^2 = 291$	$\Sigma xy = 296$

Here $n = 8$

$$r = \frac{n\Sigma xy - (\Sigma x)(\Sigma y)}{\sqrt{\left[n\Sigma x^2 - (\Sigma x)^2\right]\left[n\Sigma y^2 - (\Sigma y)^2\right]}}$$

$$= \frac{8 \times 296 - 50 \times 45}{\sqrt{\left[8.332 - (50)^2\right]\left[8.291 - (45)^2\right]}}$$

$$= \frac{2368 - 2250}{\sqrt{(2656 - 2500)(2328 - 2025)}}$$

$$= \frac{118}{\sqrt{156 \times 303}} = \frac{118}{\sqrt{47268}}$$

$$= \frac{118}{217.41} = 0.54.$$

5 (i) Karl Pearson's coefficient of correlation between two variables x and y is 0.52, their covariance is + 7.8. If the variance of x is 16, find the standard deviation of y. *[C.U. 1983]*

(ii) Find the covariance between x and y when variance of x is 6.25, variance of y is 4 and correlation coefficient of x and y is 0.18.

Ans : (i) Here $r = 0.52$, $\text{cov}(x, y) = 7.8$ and $\text{var}(x) = 16$

$\therefore$ S.D. of $x\,(\sigma_x) = \sqrt{16} = 4$

Now, $r = \dfrac{\text{cov}(x, y)}{\sigma_x \sigma_y}$

or, $0.52 = \dfrac{7.8}{4 \times \sigma_y}$, or, $\sigma_y = \dfrac{7.8}{0.52 \times 4} = \dfrac{7.8}{2.08} = 3.75$

$\therefore$ S.D. of $y = 3.75$

(ii) According to problem, $\text{var}(x) = 6.25$, $\text{var}(y) = 4$ and $r = 0.18$.

Then, $\sigma_x^2 = \text{var}(x) = 6.25$ $\quad\therefore\ \sigma_x = \sqrt{6.25} = 2.5$

and $\sigma_y^2 = \text{var}(y) = 4$ $\quad\therefore\ \sigma_y = \sqrt{4} = 2$

Now, we know that $r = \dfrac{\text{cov}(x, y)}{\sigma_x \cdot \sigma_y}$ $\quad\therefore\ 0.18 = \dfrac{\text{cov}(x, y)}{2.5 \times 2}$

or, $\text{cov}(x, y) = 0.18 \times 2.5 \times 2 = 0.9$

$\therefore$ the required covariance between x and y is 0.9.

Advantages and Disadvantages of Product Moment *r*

(*i*) *Advantages:*

(*a*) The coefficient of correlation indicates both the directions (i.e. whether the correlation is positive or negative) as well as degree of correlation.

(*b*) The values of r lie between -1 and $+1$. Absence of correlation is denoted by zero.

(*c*) It takes into account all items of the variable and is thus based on a suitable measure of variation.

(*ii*) *Disadvantages*:

(*a*) It always assumes always the linear relationship between the variables regardless of the facts that assumption is correct or not.

(*b*) It takes more time to compute as compared to other methods.

(*c*) It is not easy to interpret the significance of coefficient of correlation. Very often it is misinterpreted.

(*d*) It is effected by the extreme items.

(*e*) If the data are not reasonably homogeneous the coefficient of correlation may give a misleading picture.

Spearman's Rank Correlation Coefficient

There are some attributes in Education and Psychology (such as intelligence, leadership, honesty, morality, character, etc.) where quantitative measurement is not possible. Again, there are some problems when data exhibits some relationship but not linear correlation, and cannot be described by Pearson's correlation coefficient. In such cases, the objects or individuals may be ranked and arranged in order of merit or proficiency on two variables when these two sets of ranks have agreement between them, we measure the degree of relationship by rank correlation. For example, the evaluation of a group of students on the basis of leadership ability, the ordering of women in a beauty contest, students ranked in order of preference or the pictures may be ranked according to their asthetic values. Spearman has developed a formula called rank correlation coefficient to measure the extent or degree of correlation between two sets of ranks.

The formula to calculate the rank correlation coefficient is.

$$R = 1 - \frac{6\sum d^2}{n^3 - n}$$

Difference between paired ranks (in each case)

Number of pairs of observations

Rank correlation coefficient

TIED RANKS

- If in a series two or more items have the same score, they must be allotted the same ranks. This is the case of 'tied ranks'. We first find the average of the ranks of such items and assign the average rank to each of such items. For example, say three student have got the highest marks. So they must have the same rank and the rank for each of them will be

$$\frac{1+2+3}{3}=\frac{6}{3}=2$$

- In case of tied ranks, Spearman's Formula has been modified and it is

$$R=1-\frac{6\left[\sum d^2+\sum\frac{t^3-t}{12}\right]}{n^3-n}$$

where t is the number of items involved in a tie in both the series, t is called tie length.

While solving the rank correlation coefficient one may come across the following types of problems:

(*a*) Where actual ranks are given
(*b*) Where actual ranks are not given
(*c*) Equal ranks or tie in ranks

ILLUSTRATION 4

A. *Where actual ranks are given*

Ten competitors in a beauty contest are ranked by two judges in the following order. Calculate Spearman's rank correlation coefficient.

Competitors :	A	B	C	D	E	F	G	H	I	J
1st Judge :	1	6	5	10	3	2	4	9	7	8
2nd Judge :	6	4	9	8	1	2	3	10	5	7

[*C.U. B.Com. (G) 2016*]

Solution: **Calculation of Spearman's Rank correlation coefficient**

Competitors	1st Judge (r_1)	2nd Judge (r_2)	$d = r_1 - r_2$	d^2
A	1	6	–5	25
B	6	4	2	4
C	5	9	–4	16
D	10	8	2	4
E	3	1	2	4
F	2	2	0	0
G	4	3	1	1
H	9	10	–1	1
I	7	5	2	4
J	8	7	1	1
$n = 10$			$\Sigma d = 0$	$\Sigma d^2 = 60$

$$R = 1 - \frac{6\sum d^2}{n^3 - n}$$

$$= 1 - \frac{6 \times 60}{10^3 - 10}$$

$$= 1 - \frac{360}{990} = 1 - 0.36 = 0.64 (Approx)$$

ILLUSTRATION 5

B. *Where actual ranks are not given*

In a contest, two judges assessed the performance of the eight candidates A, B, C, D, E, F, G and H as shown in the data following:

	A	B	C	D	E	F	G	H
1st Judge :	52	67	40	72	55	48	60	43
2nd Judge :	52	51	43	54	56	40	60	49

Find the rank correlation of the above data applying Spearman's method.

[*C.U. B.Com. 1992, 2014 (H)*]

Solution: **Calculation of Rank correlation coefficient**

Candidates	1st Judge	2nd Judge	Rank in 1st Judge's assessment (r_1)	Rank in 2nd Judge's assessment (r_2)	$d = r_1 - r_2$	d^2
A	52	52	5	4	1	1
B	67	51	2	5	–3	9
C	40	43	8	7	1	1
D	72	54	1	3	–2	4
E	55	56	4	2	2	4
F	48	40	6	8	–2	4
G	60	60	3	1	2	4
H	43	49	7	6	1	1
$n = 8$					$\Sigma d = 0$	$\Sigma d^2 = 28$

$$R = 1 - \frac{6\Sigma d^2}{n^3 - n}$$

$$= 1 - \frac{6 \times 28}{8^3 - 8}$$

$$= 1 - \frac{168}{504} = 1 - 0.33 = 0.67.$$

ILLUSTRATION 6

The following data give the scores of 10 students on two trials of test with a gap of 2 weeks i.e. Trial I and Trial II. Compute the correlation between the scores of two trials by rank difference method:

Student :	A	B	C	D	E	F	G	H	I	J
Trial I :	10	15	11	14	16	20	10	8	7	9
Trial II :	16	16	24	18	22	24	14	10	12	14

Solution: **Computation of rank correlation coefficient**

Student	Trial-I (X)	Trial-II (Y)	Rank on Trial-I (r_1)	Rank on Trial-II (r_2)	$d = r_1 - r_2$	d^2
A	10	16	6.5	5.5	1	1.00
B	15	16	3	5.5	–2.5	6.25
C	11	24	5	1.5	3.5	12.25
D	14	18	4	4	0	0
E	16	22	2	3	–1	1.00
F	20	24	1	1.5	–0.5	0.25
G	10	14	6.5	7.5	–1	1.00
H	8	10	9	10	–1	1.00
I	7	12	10	9	1	1.00
J	9	14	8	7.5	0.5	0.25
$n = 10$					$\Sigma d = 0$	$\Sigma d^2 = 24$

In the above table Columns 2 and 3 show that more than one students are getting the same scores. In column 2 students A and G are getting the same score viz. 10 and they possess 6th and 7th rank in the group. So the rank of A and G will be the average of the two ranks, i.e. $\frac{6+7}{2} = \frac{13}{2} = 6.5$.

In column 3 in respect of scores on Trial II, ties occur at three places. Students C and F have the same score and hence are assigned the average rank of $\frac{1+2}{2}=\frac{3}{2}=1.5$. Students A and B have rank position 5 and 6, hence are assigned $5.5\left(\frac{5+6}{2}\right)$ rank each. Similarly students G and J have been assigned $7.5\left(\frac{7+8}{2}\right)$ rank each.

In Trial-I, $t=2$

In Trial-II, $t=2$, 2 and 2 [t = number of items whose ranks are common]

Therefore,

$$\Sigma\left(\frac{t^3-t}{12}\right)=\frac{2^3-2}{12}+\frac{2^3-2}{12}+\frac{2^3-2}{12}+\frac{2^3-2}{12}$$

$$=\frac{6}{12}+\frac{6}{12}+\frac{6}{12}+\frac{6}{12}$$

$$=\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$$

$$=2$$

Therefore, $$R=1-\frac{6\left\{\Sigma d^2+\Sigma\left(\frac{t^3-t}{12}\right)\right\}}{n^3-n}$$

$$=1-\frac{6(24+2)}{1000-10}$$

$$=1-\frac{6\times 26}{990}=1-\frac{156}{990}$$

$$=1-0.157$$

$$=0.843\,(\text{Approx.})$$

Advantages and Disadvantages of Rank Correlation Method

(i) *Advantages*:

- Computation of 'R' under this method is quicker and easier than computation of 'r' by the Pearson's Product Moment Method.
- It is quiet easy to interpret 'R'.
- It is an acceptable method if data are available only in ordinal form or number of paired variable lies between 5 and 30 with a minimum or a few ties in ranks.

(ii) *Disadvantages*

- If the number of observations are more, giving ranks to them becomes a tedious job.
- When the interval data are converted into rank-ordered data the information about the size of the score differences is lost; e.g. in Illustration-6, if D in Trial-II gets scores from 18 up to 21, his rank remain only 4.

Rank Correlation

EXAMPLE 7

Find R (Spearman Rank correlation coefficient) when $\Sigma d^2 = 30$ and $n = 10$

[***C.U. B.Com. 2001, 2012, 2014 (G), 2016 (G), 2016(H)***]

Solution: We know that,

$$R = 1 - \frac{6\Sigma d^2}{n^3 - n}$$

$$= 1 - \frac{6 \times 30}{10^3 - 10} \quad (\text{as } \Sigma d^2 = 30 \text{ and } n = 10)$$

$$= 1 - \frac{180}{990} = 1 - 0.18$$

$$= 0.82$$

Therefore, Spearman's Rank correlation coefficient is 0.82.

6.13. ASSOCIATION

Statistics deals with the quantitative (Measurable) data. But in many investigations it is required to collect data for a set of individuals (items) of a population which are qualitative in nature and possess non-measurable characteristics. In such cases where it is not possible to measure the magnitude of the variable, the investigator can only study the presence or absence of a particular quality in a group of individuals (items). These types of data with qualitative characteristics are called **attributes**. For example, religion, literacy, intelligence etc. are attributes. Though attributes cannot be measured numerically, but they can be classified into non-overlaping classes according to the presence and absence of the attributes.

For example, if we consider the attribute 'marital status', the population is divided into two classes viz., married and unmarried (not married). So, each individual of the population either belong to one class or, to another class. In this way, we can count the number of individuals belonging to each class.

If we consider two attributes, say blindness and deafness, we have four non-overlaping classes viz., blinds and deafs, blinds and nondeafs, unblinds and deafs, unblinds and nondeafs. All these four classes are sub-classes of the total population. That is, each individual of all these four classes belongs to the population. Here, we can count the number of individuals belonging to each class and that number is called the **frequency** of that class.

In statistical investigation many times we need to study the data relating to a set of individuals having more than one qualitative characteristics. For example, blindness and deafness of persons may be investigated in a community. Then it should be observed whether there is any relationship between blindness and deafness. In quantitative data **correlation** determines the nature and degree of this relationship between two variables. In qualitative data **Association** determines the nature and degree of the relationship between two attributes. So, in statistics Association is a technical term and it is used to measure the strengths of the relationship between two attributes.

6.5.2 Classification and Notations

The population in the theory of attributes is divided into two classes, namely the negative class and the positive class. The positive class signifies that the attribute is present in that particular item under study, and this class in the theory of attributes is represented by capital letters such as A, B, etc. The negative class signifies that the attribute is not present in that particular item under study, and this class in the theory of attributes is represented by the Greek letter 'α' (alpha) and 'β' (beta), etc.

Thus 'α' = not *A* and 'β' = not B. For example, if *A* represents boys, the 'α' would represents girls. Similarly, if *B* represents tall, then 'β' would denotes short. The combination of the different attributes is denoted by (AB), (Aβ), (α, β) and (α, β). Thus, in this example (AB) would mean number of tall boys and (αβ) short girls. The number of observations that have been allocated in the attributes is known as the **class frequencies**. These class frequencies are symbolically denoted by bracketing the attribute terminologies like (AB), (αβ), etc. Thus, if the number of tall boys is 20, the frequency of class (AB) is 20.

6.5.3 Order of Classes and Class Frequencies

A class represented by the 'n' attribute refers to the class that has the nth order and its frequency as a frequency of the nth order. A class having one attribute is known as the class by the first order and a class having two attributes as class of the second order, and so on. Thus when there are two attributes A and B only and N is denoted by the total number of observations (called the frequency), then N is the frequency of the zero order (A), (B), (α), (β) are the frequencies of the first order (AB), (Aβ), (αB), (αβ) are the frequencies of the second order.

In such a case, classes of the second order would also be known as the classes of the ultimate order. If N is counted as a class, then the total number of classes is always equal to 3^n, n represents the number of attributes. But the total number of classes of the ultimate order is always equal to 2^n.

If two attributes A and B are taken into consideration, then the total number of classes will be $3^2 = 9$ (A, α, B, β, AB, Aβ, αB, αβ and N) but the total number of classes of the ultimate order would be $2^2 = 4$ (i.e. AB, Aβ, αB, αβ).

The contingency table of order (2×2) for two attributes A and B can be shown as given below:

	A	**α**	**Total**
B	(AB)	(αB)	(B)
β	(Aβ)	(αβ)	(β)
Total	(A)	(α)	N

6.5.4 Relationship between the Class Frequencies

The frequency of a lower class can always be expressed in terms of class frequencies of the higher order. Thus we can write:

$$N = (A) + (\alpha) = (B) + (\beta)$$

$$(A) = (AB) + (A\beta)$$

$$(\alpha) = (\alpha B) + (\alpha\beta)$$

$$(B) = (AB) + (\alpha B)$$

$$(\beta) = (A\beta) + (\alpha\beta)$$

6.6 CONSISTENCY OF DATA

Consistency of a set of class frequencies may be defined as the property that none of these is negative and so the data are said to be consistent. It means there is a consistency between the different frequencies and there is no conflict in any way. Otherwise, the data for class frequencies are said to be 'inconsistent' i.e. if any class frequency is negative, we say that the data is inconsistent.

ILLUSTRATION 7

From the following cases find out whether the data are consistent or not:

(i) $N = 1000$, (A) = 600, (AB) = 50 and (B) = 500

(ii) $N = 60$, (A) = 51, (B) = 32 and (AB) = 25

Solution:

(i) Given: $N = 100$, (A) = 600, (AB) = 50 and (B) = 500

Substituting the values in the following table:

	A	α	Total
B	50	450	500
β	550	–50	500
Total	600	400	1000

As (αβ) is negative, then the given data is inconsistent.

(ii) Given: $N = 60$, (A) = 51, (B) = 32 and (AB) = 25

Substituting the values in the following table:

	A	α	Total
B	25	7	32
β	26	2	28
Total	51	9	60

As all frequencies are positive, then we can conclude that the given data is consistent.

METHODS OF STUDYING ASSOCIATION

The following methods may be used to study whether two attributes are associated or not:

(a) Comparison of observed and expected frequencies method

(b) Proportion method

(c) Yule’s coefficient of association

6.8.1 Comparison of Observed and Expected Frequencies Method

Under this method, the actual observation is compared with the expectation. If the actual observation is more than the expectation, the attributes are said to be positively associated; if the actual observation is less than the expectation, the attributes are said to be negatively associated and if the actual observation is equal to the expectation, the attributes are said to be independent.

Symbolically, two attributes A and B are

(*i*) positively associated

$$\text{if} \quad (AB) \quad > \quad \frac{(A)\times(B)}{N}$$

[actual observation] [expectation]

(*ii*) negatively associated

$$\text{If} \quad (AB) \quad < \quad \frac{(A)\times(B)}{N}$$

[actual observation] [expectation]

(*iii*) independent

$$\text{if} \quad (AB) \quad = \quad \frac{(A)\times(B)}{N}$$

[actual observation] [expectation]

It is also true in case of attributes α and B; α and β and A and B.

As from the following data find out whether attributes (i) (AB), (ii) (Aβ), (iii) (αB) and (iv) (αβ) are independent, associated or disassociated. Given N = 80, (A) = 32, (B) = 64, (AB) = 24.

Solution:

(i) Expectation of $(AB) = \frac{(A)\times(B)}{N} = \frac{32\times 64}{80} = 25.6$

The actual observation (AB) = 24

As actual observation (24) is less than the expectation (25.6), then, attributes are negatively associated or disassociated.

(ii) In order to find out the nature of association between the attributes (Aβ), (αB) and (αβ) we shall have to determine the unknown values by preparing a nine square table:

	A	**α**	**Total**
B	24	40	64
β	8	8	16
Total	32	48	80

From the above table we get, (Aβ) = 8, (αB) = 40, (αβ) = 8, (α) = 48, (β) = 16

Expectation of $(A\beta) = \frac{(A)\times(\beta)}{N} = \frac{32\times 16}{80} = 6.4$

The actual observation (Aβ) = 8

As the actual observation (8) is more than the expectation (6.4), then the attributes *A* and β are positively associated.

(iii) Expectation of $(\alpha B) = \frac{(\alpha)\times(B)}{N} = \frac{48\times 64}{80} = 38.4$

The actual observation (αB) = 40

As the actual observation (40) is more than the expectation (38.4), then the attributes α and *B* are positively associated.

(iv) Expectation of $(\alpha\beta) = \frac{(\alpha)\times(\beta)}{N} = \frac{48\times 16}{80} = 9.6$

The actual observation (αβ) = 8

As the actual observation (8) is less than the expectation (9.6), then the attributes α and β are negatively associated or disassociated.

[This method can not measure the degree of association between the attributes. It can only measure the nature of association.]

6.8.2 Proportion Method

If there is no relationship of any kind between two attributes, we define them independent and we find the same proportion of A's amongst the B's as amongst the β's. for example, if blindness and dumbness are not associated, then the proportion of blind people amongst the dumb and amongst the not-dumbs must be equal.

Symbolically, two attributes A and B are

(*i*) positively associated if $\frac{(AB)}{(B)} > \frac{(A\beta)}{(\beta)}$

(*ii*) negatively associated if $\frac{(AB)}{(B)} < \frac{(A\beta)}{(\beta)}$

(*iii*) independent if $\frac{(AB)}{(B)} = \frac{(A\beta)}{(\beta)}$

If the relation (iii) holds good, the corresponding relations

$$\frac{(\alpha B)}{(B)} = \frac{(\alpha\beta)}{(\beta)};\ \frac{(AB)}{(A)} = \frac{(\alpha\beta)}{(\alpha)};\ \frac{(AB)}{(A)} = \frac{(\alpha\beta)}{(\alpha)}$$

must also hold true.

In a population of 250 people the number of married is 100. Out of 75 people who are unemployed 30 belonged to the married group. It is required to find out whether the attributes marriage and unemployed are independent, positively associated or negatively associated.

Solution: Let A denote married people.

$\therefore$ α represents unmarried people.

Let B devote number of unemployed people

$\therefore$ β would denote employed people.

We are given the total number of people, i.e. $N = 250$.

(A) = 100, (B) = 75 and (AB), i.e. the number of married people who are unemployed = 30.

$\therefore$ Proportion of married people who are unemployed:

$$\text{i.e., } \frac{(AB)}{(A)} = \frac{30}{100} = 0.3 \text{ or } 30\%$$

and proportion of unmarried people who are unemployed:

$$\text{i.e., } \frac{(\alpha B)}{(\alpha)} = \frac{45}{150} = 0.3 \text{ or } 30\%$$

Nine square table

	A	**α**	**Total**
B	30	45	75
β	70	105	175
Total	100	150	250

As the two proportions are the same, then we conclude that the attributes, marriage and unemployed, are independent.

[Like previous method, this method also can not measure the degree of association. It can only measure the nature of association.]

6.8.3 Yule's Coefficient of Association

It is the most popular and superior method of studying association of attributes. This method not only measures the nature of association, but also the degree or extent to which the two attributes are associated. To find out the degree of association between two attributes Yule has defined coefficient of association Q by the formula,

$$Q = \frac{(AB)(\alpha\beta) - (A\beta)(\alpha B)}{(AB)(\alpha\beta) + (A\beta)(\alpha B)} \tag{6.8}$$

This is a relative measure of association. The absolute measure of association is given by

$$(AB)(\alpha\beta) - (A\beta)(\alpha B) = 0$$

$$\text{or } (AB)(\alpha\beta) = (A\beta)(\alpha B)$$

The coefficient of association varies between +1 and −1. When the value of Q is +1 there is complete association, i.e. perfect positive association between the attributes, because in that case the second term in both the numerator and denominator of the value of Q is Zero. When the value of Q is −1 there is perfect dissociation, i.e. perfect negative association between the attributes, because in that case the first term in both the numerator and denominator is zero. When the value of Q is zero, the two attributes are independent, i.e. there is no association.

Find the Association between Literacy and Unemployment from the following figures:

Total adults	10,000
Literates	1,290
Unemployed	1,390
Literate unemployed	820

Comment on the results.

Solution: Let A denote literates

$\therefore$ α will denote illiterates

Let B denote unemployed

$\therefore$ β will denote employed

Given: $(A) = 1290$, $(B) = 1390$, $(AB) = 820$, $N = 10000$

By putting the known values in the nine square table, we can find out the unknown values.

	A	α	Total
B	820	570	1,390
β	470	8140	8,610
Total	1290	8710	10,000

$(\alpha B) = (B) - (AB)$

$= 1390 - 820 = 570$

$(A\beta) = (B) - (AB)$

$= 1290 - 820 = 470$

$(\alpha) = N - (A)$

$= 10000 - 1290 = 8710$

$(\alpha\beta) = (\alpha) - (\alpha B)$

$= 8710 - 570 = 8140$

$$Q = \frac{(AB)(\alpha\beta) - (A\beta)(\alpha B)}{(AB)(\alpha\beta) + (A\beta)(\alpha B)}$$

$$= \frac{(820 \times 8140) - (470 \times 570)}{(820 \times 8140) + (470 \times 570)}$$

$$= \frac{6674800 - 267900}{6674800 + 267900} = \frac{6406900}{6942700} = 0.923$$

Therefore, there is a high degree of positive association between literacy and unemployment.

21. Given $r = 0.8$, $\Sigma xy = 80$, $\sigma_x = 2$, $\Sigma y^2 = 100$, where $x = X - \overline{X}$ and $y = Y - \overline{Y}$; the number of items will be

(*a*) 12 (*c*) 20
(*b*) 15 (*d*) 25 [**Ans.** (*d*)]

22. If the two variables are independent of each other, then value of '*r*' is

(*a*) 1 (*b*) 2 (*c*) 0 (*d*) 0.5 [**Ans.** (*c*)]

23. The value of the coefficient of correlation lies between

(*a*) ±1 (*b*) ±2 (*c*) ±3 (*d*) ±1.5 [**Ans.** (*a*)]

24. What is the covariance if the coefficient of correlation between *x* and *y* is 0.87 and the variance of *x* and *y* are 36 and 25 respectively

(*a*) 18.25 (*b*) 26.10 (*c*) 19.25 (*d*) 21.6 [**Ans.** (*b*)]

25. When we conduct a study that examines the relationship between two variables, then we are working with

(*a*) Univariate data (*c*) Bivariate data
(*b*) Multivariate data (*d*) None of there [**Ans.** (*c*)]

Rank Correlation

26. If $\Sigma d^2 = 33$ and $n = 10$, where d represents the difference between the rank of two series and *n* is the number of pairs of observations, then the value of the coefficient of rank correlations will be [***C.U. B.Com. 1985***]

(*a*) 0.68 (*c*) 0.75
(*b*) 0.7 (*d*) 0.8 [**Ans.** (*d*)]

27. The value of R (Spearman Rank correction coefficient) when $\Sigma d^2 = 30$ and $n = 10$, is

(*a*) 0.75 (*c*) 0.82
(*b*) 0.65 (*d*) 0.9
[***C.U. B.Com. 2001, 2012, 2014 (G), 2016 (G), 2016 (H)***] [**Ans.** (*c*)]

28. The following are the ranks of 10 students in English and Maths

Sr. no. :	1	2	3	4	5	6	7	8	9	10
Rank in Math :	9	6	4	5	10	3	1	7	2	8
Rank in English :	8	9	3	6	7	1	2	5	5	10

The coefficient of rank correlation between the marks in Maths and English is

(*a*) 0.61 (*c*) 0.59
(*b*) 0.769 (*d*) 0.79 [**Ans.** (*b*)]

29. In case of tie in ranks, the rank of the tied scores are calculated by using

(*a*) Mean (*c*) Mode
(*b*) Median (*d*) S.D [**Ans.** (*a*)]

30. Differences of the ranks (d) are squared to remove

(*a*) deviations (*c*) positive values
(*b*) negative values (*d*) none of there [**Ans.** (*b*)]

Find *R* (Spearman Rank correlation coefficient) when $\Sigma d^2 = 30$ and $n = 10$

[***C.U. B.Com. 2001, 2012, 2014 (G), 2016 (G), 2016(H)***]

Solution: We know that,

$$R = 1 - \frac{6\Sigma d^2}{n^3 - n}$$

$$= 1 - \frac{6 \times 30}{10^3 - 10} \quad (\text{as } \Sigma d^2 = 30 \text{ and } n = 10)$$

$$= 1 - \frac{180}{990} = 1 - 0.18$$

$$= 0.82$$

Therefore, Spearman's Rank correlation coefficient is 0.82.

Association of Attributes

31. A qualitative characteristic is called
    (*a*) constant (*c*) attribute
    (*b*) variable (*d*) association **[Ans. (*c*))**

32. If an attribute has two classes, it is called
    (*a*) Trichotomy
    (*b*) Dichotomy
    (*c*) Simple classification
    (*d*) Manifold classification **[Ans. (*b*)]**

33. With two attributes *A* and *B*, the total number of ultimate frequencies is
    (*a*) Two (*c*) Six
    (*b*) Four (*d*) Nine **[Ans. (*b*)]**

34. If $(AB) = \frac{(A)(B)}{N}$, the two attributes *A* and *B* are
    (*a*) Independent (*c*) Correlated
    (*b*) Dependent (*d*) Quantitative **[Ans. (*a*)]**

35. If the class frequency (AB) = 0, the value of Q is equal to
    (*a*) 0 (*c*) −1
    (*b*) 1 (*d*) 0 to 1 **[Ans. (*c*)]**

36. If for two attributes the class frequencies are (AB) (αβ) = (Aβ) (αB), then Q is equal to:
    (*a*) 0 (*c*) +1
    (*b*) −1 (*d*) α **[Ans. (*a*)]**

37. If two attributes *A* and *B* are independent, then the coefficient of association is:
    (*a*) −1 (*c*) 0
    (*b*) +1 (*d*) 0.5 **[Ans. (*c*)]**

38. If $(AB) < \frac{(A)(B)}{N}$, the association between two attributes *A* and *B* is;
    (*a*) Positive (*c*) Symmetrical
    (*b*) Zero (*d*) Negative **[Ans. (*d*)]**

39. If (AB) (αβ) > (Aβ) (αB), then *A* and *B* are said to be
    (*a*) Negatively associated
    (*b*) Positively associated
    (*c*) Independent
    (*d*) Difficult to tell **[Ans. (*b*)]**

40. If two attributes *A* and *B* have perfect positive association, the value of coefficient of association is equal to
    (*a*) 0 (*c*) (r − 1) (c − 1)
    (*b*) −1 (*d*) +1 **[Ans. (*d*)]**

Association of Attributes

21. Given N = 2,000, (A) = 1,500, (B) = 100, (AB) = 350. Are the data consistent?
(*a*) Yes (b) No [**Ans.** (*b*)]

22. Given N = 280, (A) = 250, (B) = 85, (AB) = 35. Are the data consistent?
(*a*) Yes (*c*) Cannot say [**Ans.** (*b*)]
(*b*) No

23. In a sample of 1,000 individuals, 100 possess the attribute A and 300 possess attribute B. If A and B are independent, how many individual possess A and B?
(*a*) 28 (*c*) 30
(*b*) 25 (*d*) 35 [**Ans.** (*c*)]

21.

	A	α	Total
B	350	-250	100
β	1150	750	1900
Total	1500	500	2000

22.

	A	α	Total
B	35	-50	85
β	215	-20	195
Total	250	30	280

23.

	A	α	Total
B	30	270	300
β	70	630	700
Total	100	900	1000

EXAMPLE 8

From the following data show that attributes α and β are independent.
(α) – 125, (β) – 45, (αβ) – 25 and N – 225

Solution: Expectation of $(\alpha\beta) = \frac{(\alpha)\times(\beta)}{N} = \frac{125\times 45}{225} = 25$

The actual observation (αβ) = 25

As the actual observation (25) is equal to the expectation (25), then the attributes α and β are independent.

EXAMPLE 9

From the following case find out whether the data are consistent or not:
$(A) = 100$, $(B) = 150$, $(AB) = 140$, $N = 500$

Solution: We know that,
$(A\beta) = (A) - (AB) = 100 - 140 = -40$
As (AB) is negative, then the given data is inconsistent.

EXAMPLE 10

Using proportion method, determine the nature of association between A and B:
$(AB) = 30$, $(A) = 80$, $(\alpha B) = 20$, $(\alpha) = 120$

Solution: $\frac{(AB)}{(A)} = \frac{30}{80} = 0.375$

and $\frac{(\alpha B)}{\alpha} = \frac{20}{120} = 0.167$

Since, $\frac{(AB)}{(A)} > \frac{(\alpha B)}{\alpha}$ the attributes A and B are positively associated.

REGRESSION ANALYSIS

REGRESSION

Regression is a statistical method depending upon a given bivariate data to estimate the unknown values of one variable (called dependent variable) for specified values of the other variable (called independent variable) by establishing functional relationship between the variables.

According to M. M. Blair, “Regression is the measure of the average relationship between two or more variables in terms of the original units of the data.”

According to T. Yamane “one of the most frequently used technique in economics and business research, to find a relation between two or more variables that are related causally is regression analysis.”

REGRESSION ANALYSIS

Regression analysis is a statistical tool which investigates the relationship between variables in the form of mathematical equations. It is the most useful form of analysis as it studies the variables individually and determines their significance with greater accuracy. Say, for example, you want to find out the effect of price increase on the customer’s demand for fixing selling price of a product or you want to study the relationship between salaries and qualifications on the job performance of an employee. These studies will naturally involve a lot of co-related variables that will individually have an effect on the dependent variable. These complex questions can be easily answered with the help of regression analysis. In short, a good regression analysis needs sound reasoning and proper interpretation of data for highly accurate predictions, forecast and solutions.

TYPES OF REGRESSION TECHNIQUES

Regression techniques basically involve the assembling of data on the variables under study and estimating the quantitative effect of a variable on another. There are different kinds of regression techniques such as:

1. Linear Regression:

This is a simple and easy to use method that models the relationship between a scalar dependent variable y and one or more explanatory (independent) variables denoted as x. Usually, more than one independent variable influences the dependent variable. When one independent variable is used in a regression, it is called a Simple Linear Regression; when two or more independent variables are used, it is called a Multiple Linear Regression. This method uses linear predictor functions for data modelling wherein unknown parameters are estimated from the data. Since linear models are linearly dependent on unknown parameters, they are easier to fit than non-linear models and lead to easier determination of statistical parameters. Multiple regression analysis introduces several additional complexities but may produce more realistic results then simple regression analysis.

2. Least Squares Method:

The typical procedure for finding the line of best fit is called the least-squares method. This method is a commonly used method to solve for linear regression equations. It’s best suited for data fitting applications such as fitting a straight line on to the points i.e. closeness to all the points in a scatter diagram etc.

The criteria for what we mean by ‘closeness’ is called the least square principle. Recall the discussion on variance, where we learnt how the variance squares the deviations around the mean. In regression we will square the deviations around the regression line instead of around the mean. The best fit regression line that has the smallest value for the squared deviations around it, the least squared deviations. That’s essentially the whole idea of least squares. Therefore, the least square method is based upon the principle that the sum of the squared residuals (residual is nothing but the difference between the actual or observed y value and y value calculated by the linear regression line equation) should be made as small as possible so the regression line has the least error. This method can be used for linear as well as non-linear regression depending on the nature of the residuals and equations.

3. Non-linear Regression:

When the relationship between variable are represented by curved lines, then it is called non-linear regression. The non-linear regression analysis uses the method of successive approximations. Here, the data are modeled by a function, which is a non-linear combination of model parameters and depends on one or more explanatory (independent) variables. Therefore, in non-linear regression too, the models could be based on simple or multiple regressions. This method takes into account the nature of relationship between the variables and tries to find some kind of transformation in them so that the relationship can be expressed easily as a straight line.

REGRESSION LINE

In scatter diagram, if the potted points corresponding to bivariate data shown linear pattern, then we may draw a best fitting straight line which gives the best average value of one variable when specific value of one variable is given. This line called Regression line.

Let there be two series viz. x-series and y-series, then there can be two lines of regression.

(1) A line which may give the best possible average values of y for given values of x is called the Regression line of y on x.

(2) A line which may give the best possible average values of x for given values of y is called the Regression line of x on y.

Thus there are two line of regression

(a) Regression line of y on x.

(b) Regression line of x on y:

(1) Regression line of y on x: This give the most probable values of y from the given values of x.

(2) Regression line of x on y: This gives the most probable values of x from the given values of y. The algebraic expression of these regression lines is called as Regression Equations. There will be two regression equations for the two regression lines.

(i) Regression equation of y on x : $y = a + bx$

(ii) Regression equation of x on y : $x = a + by$

Characteristic Of Regression Line

- If the two line of Regression coincide, then the correlation between the variables is perfect.
- If the two lines meet each other at right angles, the correlation is zero.
- The greater is the angle between the lines, the lesser is the correlation between the variables.

Index Number

WHAT IS AN INDEX NUMBER

An index number is a statistical device for measuring changes in the magnitude of a group of related variables.

Base Period

Of the two periods, the period with which the comparison is to be made, is known as the base period. The value in the base period is given the index number 100.

- Price index numbers measure and permit comparison of the prices of certain goods.

- Quantity index numbers measure the changes in the physical volume of production, construction or employment.

CONSTRUCTION OF AN INDEX NUMBER

- The Aggregative Method
- weighted aggregative price index (Laspeyre's price index)
- weighted aggregative price index (Paasche's price index)
- Method of Averaging relatives
- weighted price relative index

Commodity	Base period price (Rs) p_0	Current period price (Rs) p_1
A	2	4
B	5	6
C	4	5
D	2	3

Commodity	Base period		Current period	
	price (Rs) p_0	Quantity q_0	price (Rs) p_1	Quantity q_1
A	2	10	4	5
B	5	12	6	10
C	4	20	5	15
D	2	15	3	10

- The Aggregative Method

$$P_{01} = \frac{\sum p_1}{\sum p_0} \times 100$$

$$Q_{01} = \frac{\sum q_1}{\sum q_0} \times 100$$

Comm	p_0	p_1	q_0	q_1
A	2	4	10	5
B	5	6	12	10
C	4	5	20	15
D	2	3	15	10

- Weighted aggregative index

Laspeyre's index

$$P_{01} = \frac{\sum p_1 q_0}{\sum p_0 q_0} \times 100$$

$$Q_{01} = \frac{\sum q_1 p_0}{\sum q_0 p_0} \times 100$$

Comm	p_0	p_1	q_0	q_1
A	2	4	10	5
B	5	6	12	10
C	4	5	20	15
D	2	3	15	10

- weighted aggregative index

Paasche's index

$$P_{01} = \frac{\sum p_1 q_1}{\sum p_0 q_1} \times 100$$

$$Q_{01} = \frac{\sum q_1 p_1}{\sum q_0 p_1} \times 100$$

Comm	p_0	p_1	q_0	q_1
A	2	4	10	5
B	5	6	12	10
C	4	5	20	15
D	2	3	15	10

- The Aggregative Method

1. Edgeworth–Marshall's Formula:

$$P_{01} = \frac{\sum p_1\left(\frac{q_0 + q_1}{2}\right)}{\sum p_0\left(\frac{q_0 + q_1}{2}\right)} \times 100$$

$$Q_{01} = \frac{\sum q_1\left(\frac{p_0 + p_1}{2}\right)}{\sum q_0\left(\frac{p_0 + p_1}{2}\right)} \times 100$$

Comm	p_0	p_1	q_0	q_1	$\frac{q_0 + q_1}{2}$
A	2	4	10	5	
B	5	6	12	10	
C	4	5	20	15	
D	2	3	15	10	

- The Aggregative Method

2. Fisher's Ideal Formula:

G.M. of Laspeyre's and Paasche's index

$$= \sqrt{Laspeyre's\ price\ index \times Paasche's\ index}$$

- The Aggregative Method

3. Bowley's Formula:

A.M. of Laspeyre's and Paasche's index

$$= \frac{1}{2} \left[\text{Laspeyre's index} + \text{Paasche's index} \right]$$

- Method of Averaging relatives

$$P_{01} = \frac{1}{n}\sum \frac{p_1}{p_0} \times 100$$

$$Q_{01} = \frac{1}{n}\sum \frac{q_1}{q_0} \times 100$$

Comm	p_0	p_1	Price relative
A	2	4	
B	5	6	
C	4	5	
D	2	3	

- weighted price relative index

$$P_{01} = \frac{1}{n} \sum \left(W \times \frac{p_1}{p_0} \right) \times 100$$

Comm	Weight in % W	p_0	p_1	Price relative
A	40	2	4	
B	30	5	6	
C	20	4	5	
D	10	2	3	

Time reversal test : $I_{01} \times I_{10} = 1$ (without factor 100)

satisfied by (i) Simple aggregative formula
(ii)Edgeworth Marshall's formula
(iii) Fisher's ideal formula
(iv) Simple and weighted G.M. of price relative formula.

Factor reversal test : $P_{01} \times Q_{01} = \frac{\sum p_1 q_1}{\sum p_0 q_0}$ (value ratio)

satisfied **only** by Fisher's ideal formula.

Circular test : $I_{01} \times I_{12} \times I_{23} \times \ldots\ldots \times I_{(n-1),n} \times In_{,0} = 1$

satisfied by : (i) simple aggregative index number formula
(ii) simple G.M. of relatives formula

PROBABILITY

Chapter 3: The basic concepts of probability

Experiment: a measurement process that produces quantifiable results
(e.g. throwing two dice, dealing cards, at poker,
measuring heights of people, recording proton-proton collisions)

Outcome: a single result from a measurement
(e.g. the numbers shown on the two dice)

Sample space: the set of **all possible** outcomes from an experiment
(e.g. the set of all possible five-card hands)

The number of all possible outcomes may be
(a) **finite** (e.g. all possible outcomes from throwing a single die;
all possible 5-card poker hands)
(b) **countably infinite** (e.g. number of proton-proton events to be made before
a Higgs boson event is observed)
or (c) **constitute a continuum** **(**e.g. heights of people)

In case (a), the sample space is said to be **finite**
in cases (a) and (b), the sample space is said to be **discrete**
in case (c), the sample space is said to be **continuous**

In this chapter we consider discrete, mainly finite, sample spaces

An **event** is any subset of a sample set (including the empty set, and the whole set)

Two events that have no outcome in common are called **mutually exclusive** events.

In discussing discrete sample spaces, it is useful to use **Venn diagrams** and basic set-theory. Therefore we will refer to the **union** (A U B)**, intersection,** (A ∩ B) and **complement** ($\bar{A}$) of events A and B. We will also use set-theory relations such as

$\overline{\mathrm{A\ U\ B}} = \overline{\mathrm{A}} \cap \overline{\mathrm{B}}$ (Such relations are often proved using Venn diagrams)

This is also called De Morgan's law, another half of De Mogan's law is:

$\overline{A \cap B} = \bar{A} \cup \bar{B}$

The **classical definition of probability (classical probability concept)** states:

If there are m outcomes in a sample space (universal set), and all are equally likely of being the result of an experimental measurement, then the probability of observing an event (a subset) that contains s outcomes is given by $\frac{s}{m}$

From the classical definition, we see that the ability to count the number of outcomes in an event, and the number of outcomes in the entire sample space (universal set) is of critical importance.

The **classical definition of probability**

If there are m outcomes in a sample space, and all are equally likely of being the result of an experimental measurement, then the probability of observing an event that contains s outcomes is given by $\frac{s}{m}$

e.g. Probability of drawing an ace from a deck of 52 cards.
sample space consists of 52 outcomes.
desired event (ace) is a set of 4 outcomes (number of desired outcomes is 4)
therefore the probability of getting an ace is 4/52 = 1/13 ≈ 0.0769 (7.69%)

e.g. There are 10 motors, two of which do not work and eight which do.

a) what is the probability of picking 2 working motors
   (8!/2!/6!)/(10!/2!/8!)=(8!2!8!)/(10!2!6!)=(8!8!)/(10!6!)
b) what is the probability of picking 1 working and 1 non-working motors
c) S=2*8, m=10!/2!/8! P=16*2!*8!/10!

Mathematically, in defining probabilities of events we are deriving a set function on a sample space. A set function assigns to each subset in the sample space a real number.

Example: Consider the set function that assigns to each subset (event) *A* the number *N*(A) of outcomes in the set. This set function is additive, that is, if two events *A* and *B* have no outcomes in common (are mutually exclusive), then N(A U B) = N(A) + N(B).

Counter-example: Fig. 3.7

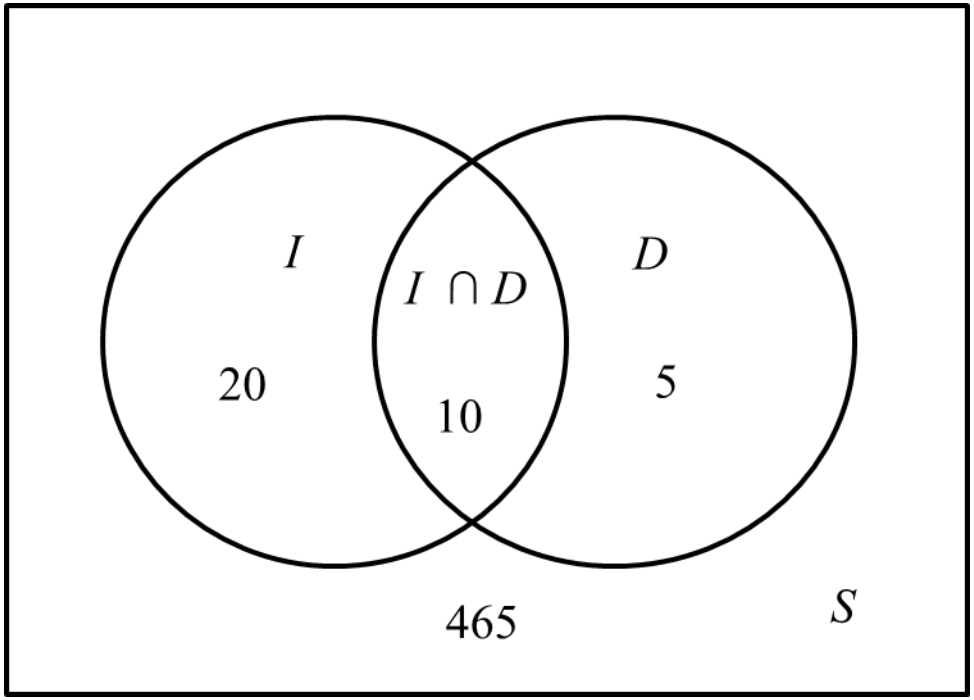


Fig. 3.7. Measurements on 500 machine parts
I = incompletely assembled, D = defective, S = satisfactory

$N(I \cup D) \neq N(I) + N(D)$ as N and D are not mutually exclusive

The axioms of probability

Let *S* be a finite sample space, *A* an event in *S*. We define *P*(*A*), the probability of *A*, to be the value of an additive set function *P*( ) that satisfies the following three conditions

Axiom 1 $0 \leq P(A) \leq 1$ for each event *A* in *S*

(probabilities are real numbers on the interval [0,1])

Axiom 2 $P(S) = 1$

(the probability of some event occurring from *S* is unity)

Axiom 3 If *A* and *B* are mutually exclusive events in *S*, then

$$P(A \text{ U } B) = P(A) + P(B)$$

(the probability function is an additive set function)

Note: these axioms do not tell us what the set function *P*(A) is, only what properties it must satisfy

The classical definition of probability defines the probability function as

$P(A) = \frac{N(A)}{N(S)}$ for any event *A* in the sample space *S*

Note that this definition satisfies all three axioms

Elementary properties of probability functions

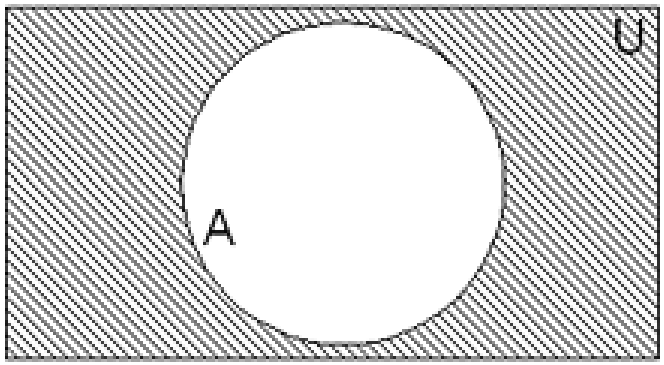
Theorem 3.4. If $A_1, A_2, \ldots, A_n$ are mutually exclusive events in a sample space *S*, then by induction on Axiom 3,

$$P(A_1 \text{ U } A_2 \text{ U } \ldots \text{U } A_n) = P(A_1) + P(A_2) + \ \ldots + P(A_n)$$

Theorem 3.7. If A is an event in S, then

$$P(\bar{A}) = 1 - P(A)$$

Proof: $P(A) + P(\bar{A}) = P(A \cup \bar{A}) = P(S) = 1$



Example: drawing 5 card with at least one spade

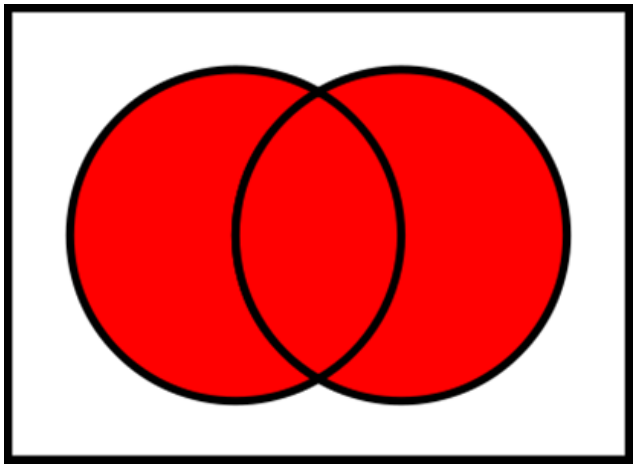
N(A) is very difficult to count, but N($\bar{A}$) is easier,

$$P(A) = 1 - P(\bar{A}) = 1 - \frac{\binom{39}{5}}{\binom{52}{5}}$$

Elementary properties of probability functions

Theorem 3.6. If A and B are any (not necessarily mutually exclusive) events in S, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Example: Find the probability of drawing 5 card with one spade or one club.

A: drawing 5 cards with one spade; B: drawing 5 card with one club.

N(A) = $\binom{13}{1}\binom{39}{4}$, N(B) = $\binom{13}{1}\binom{39}{4}$, N(A∪B) = $\binom{13}{1}\binom{13}{1}\binom{26}{3}$

P(A∪B) = $\dfrac{\binom{13}{1}\binom{39}{4}+\binom{13}{1}\binom{39}{4}-\binom{13}{1}\binom{13}{1}\binom{26}{3}}{\binom{52}{5}}$

Conditional Probability.

The probability of an event is only meaningful if we know the sample space *S* under consideration.

The probability that you are the tallest person changes if we are discussing being the tallest person in your family, or the tallest person in this class.

This is clarified using the notation $P(A|S)$, the **conditional probability** of event *A* relative to the sample space *S*.

(When *S* is understood we simply use $P(A)$)

e.g. (using classical probability)

From Fig. 3.7,

$$P(D) = P(D|S) = \frac{N(D)}{N(S)} = \frac{N(D \cap S)}{N(S)} = \frac{10+5}{500} = \frac{3}{100} = 0.03$$

$$P(D|I) = \frac{N(D \cap I)}{N(I)} = \frac{10}{30} = \frac{1}{3} = 0.33\overline{3}$$

Note:

$$P(D|I) = \frac{\frac{N(D \cap I)}{N(S)}}{\frac{N(I)}{N(S)}} = \frac{P(D \cap I)}{P(I)}$$

Conditional Probability.

If A and B are any events in S and $P(B) \neq 0$, the conditional probability of A relative to B (i.e. A often stated 'of A given B') is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad \text{(CP)}$$

From the definition of conditional probability (CP) we see that

$$P(B|A) = \frac{P(B \cap A)}{P(A)} \qquad (*)$$

Since $P(B \cap A) = \mathrm{P(A \cap B)}$, we have from (CP) and (*)

Theorem 3.8. If A and B are events in S, then

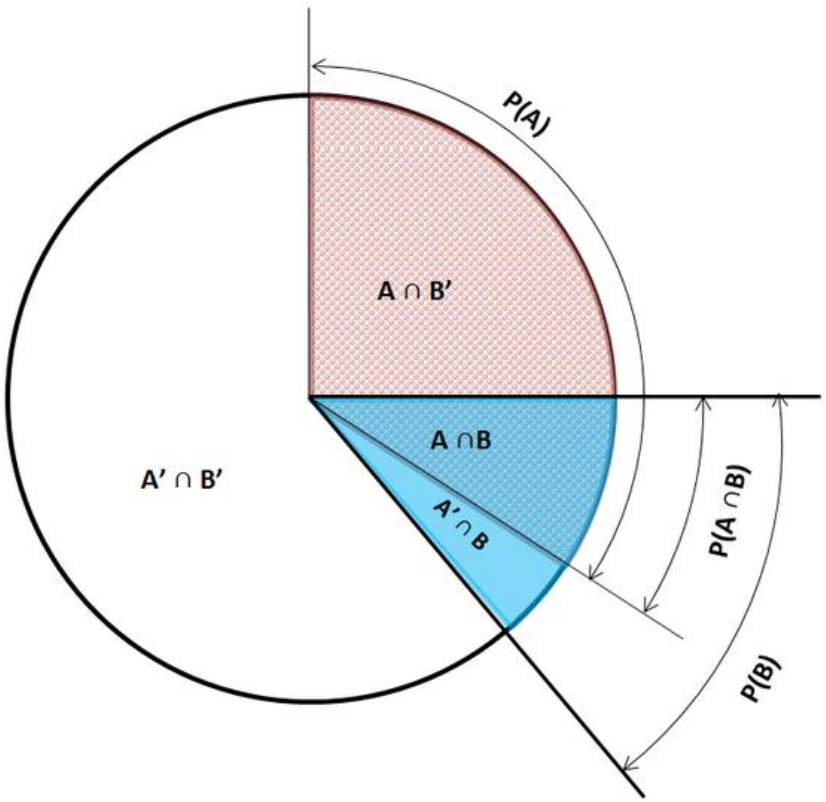
$$P(A \cap B) = \begin{cases} P(A) \cdot P(B|A) \ \text{ if } \mathrm{P(A)} \neq 0 & \text{(from } (*)) \\ P(B) \cdot P(A|B) \ \text{ if } \mathrm{P(B)} \neq 0 & \text{(from (CP))} \end{cases}$$

Theorem 3.8 is the **general multiplication rule of probability**

Theorem 3.8 is a rather shocking statement. The definition (CP) of conditional probability implies that we compute P(A|B) by knowing P(A∩B) and P(B). However, Theorem 3.8 implies we can compute P(A∩B) by knowing P(A|B) and P(B).

This implies that we often have another means at hand for computing P(A|B) rather than the definition (CP) !! (See next example)

The Venn Diagram on Conditional probability

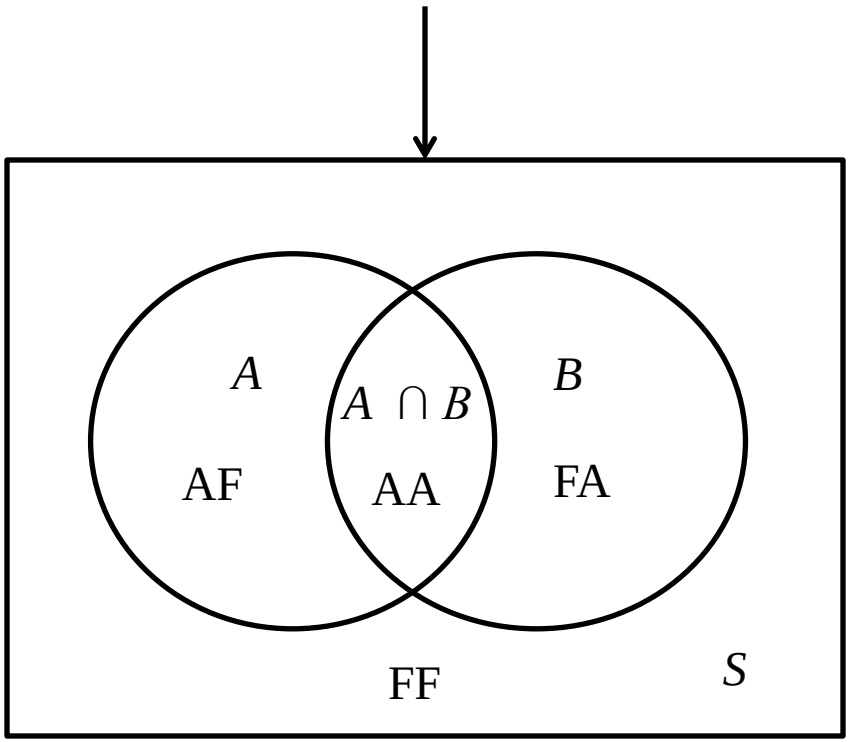


e.g. Use of the general multiplication rule

20 workers, 12 are For, 8 are Against. What is the probability of randomly picking 2 workers that are Against? (Assume classical probability).

There are 4 classes out outcomes for the 2-picks: FF, FA, AF, AA

A diagram of the sample space of all 2 picks is



Set A: all outcomes where first worker is A

Set B: all outcomes where second worker is A

Desire $P(A \cap B) = P(A) \cdot P(B|A)$

$P(A)$ = the probability that the first is 'against'
= probability of picking one 'against' from the 20 workers
= N(against)/N(workers) = 8/20

$P(B|A)$ = the probability that the second is against given that the first pick is against
= probability of picking one 'against' from 19 workers (1 'against' removed)
= N(against)/N(workers) = 7/19

Therefore $P(A \cap B) = \frac{8}{20} \cdot \frac{7}{19} = \frac{14}{95}$

If we find that $\boldsymbol{P(A|B) = P(A)}$**,** then we state that event *A* is **independent** of event *B*

We will see that event *A* is independent of event *B iff* event *B* is independent of event *A.*

It is therefore customary to state that ***A* and *B* are independent events.**

Theorem 3.9.

Two events *A* and *B* are independent events *iff* $P(A \cap B) = P(A) \cdot P(B)$

Proof:

→ If *A* and *B* are independent, that is $P(B|A) = P(B)$

Then, by Theorem 3.8,

$$P(A \cap B) = P(A) \cdot P(B|A) = P(A) \cdot P(B)$$

← If $P(A \cap B) = P(A) \cdot P(B)$

Then, by definition of conditional probability,

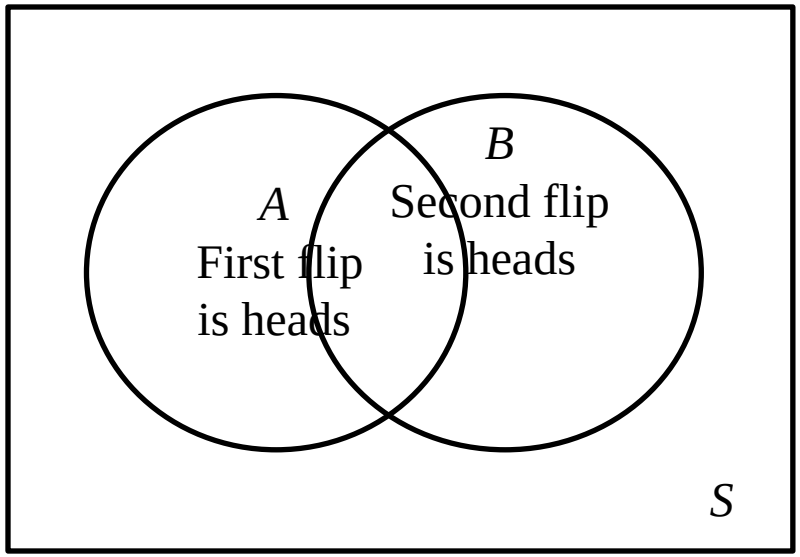
$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \cdot P(B)}{P(B)} = P(A)$$

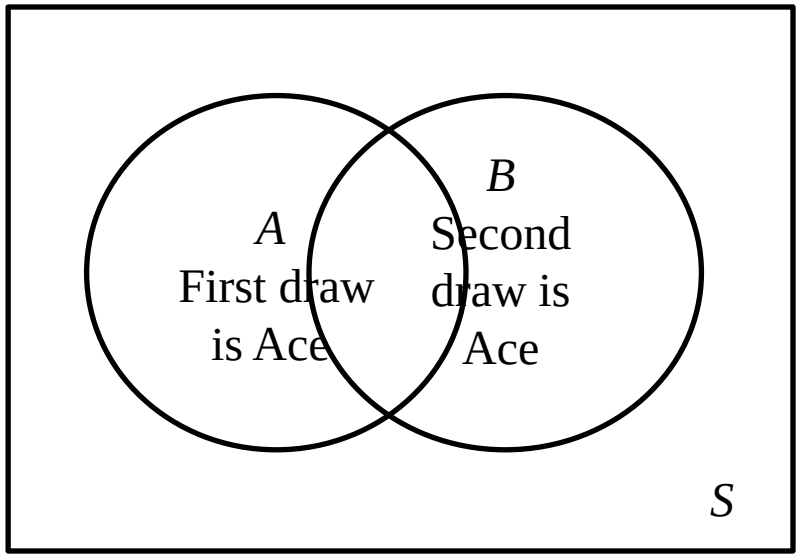
Theorem 3.9 is the **special product rule of probability** and states that **the probability that two independent events will both occur is the product of the probabilities that each alone will occur.**

Example: probability of getting two heads in two flips of a *balanced* coin

(*Assumption is* that *balance implies* that the two flips are independent)

Therefore $P(A \cap B) = P(A) \cdot P(B)$





Example: probability of selecting two aces at random from a deck of cards if first card replaced before second card drawn

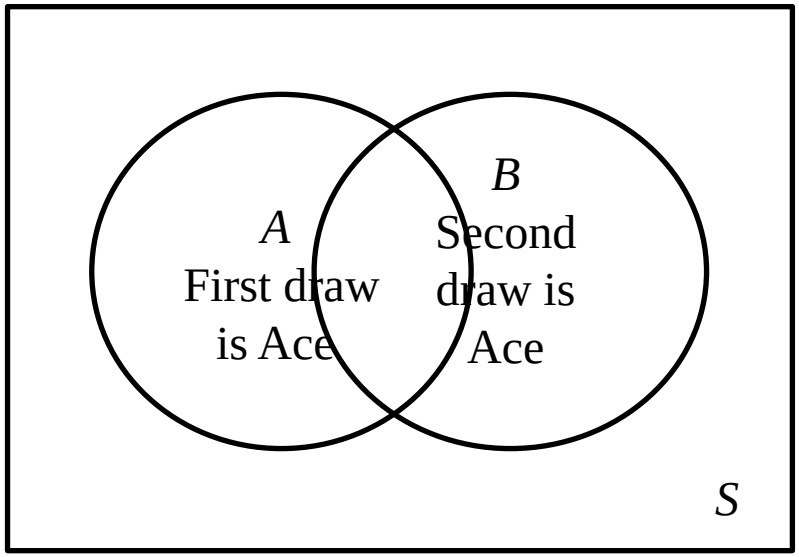
(*Assumption is* that replacing first card returns deck to original conditions making the two draws independent of each other)

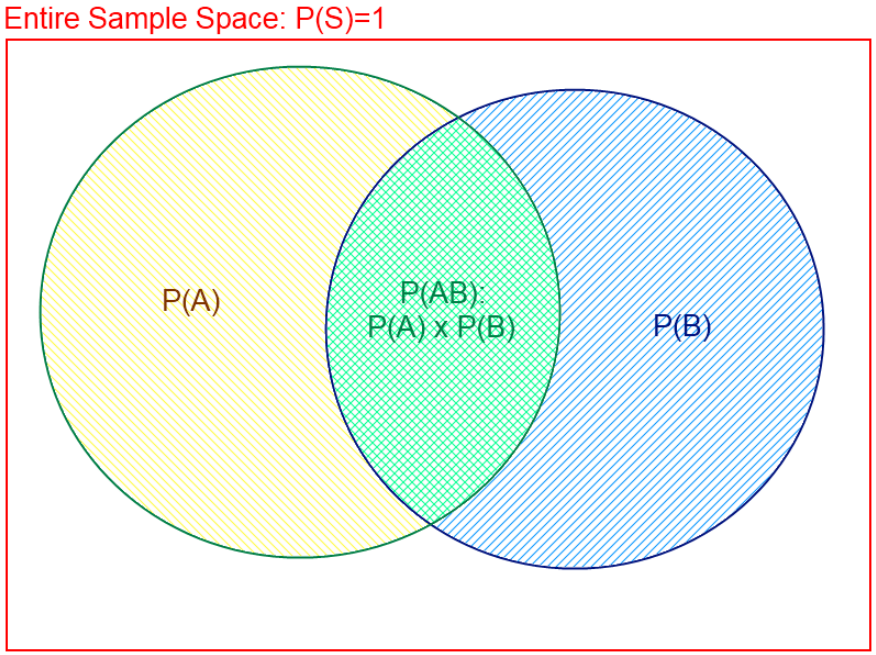
Therefore $P(A \cap B) = P(A) \cdot P(B)$

Example: probability of selecting two aces at random from a deck of cards if first card ***not*** replaced before second card drawn

(Picking the second card is now dependent on the first card choice)

Therefore $P(A \cap B) = P(A) \cdot P(B|A)$





Example: (false positives) 1% probability of getting a false reading on a test

Assuming that each test is independent of the others:

(a) probability of two tests receiving an accurate reading

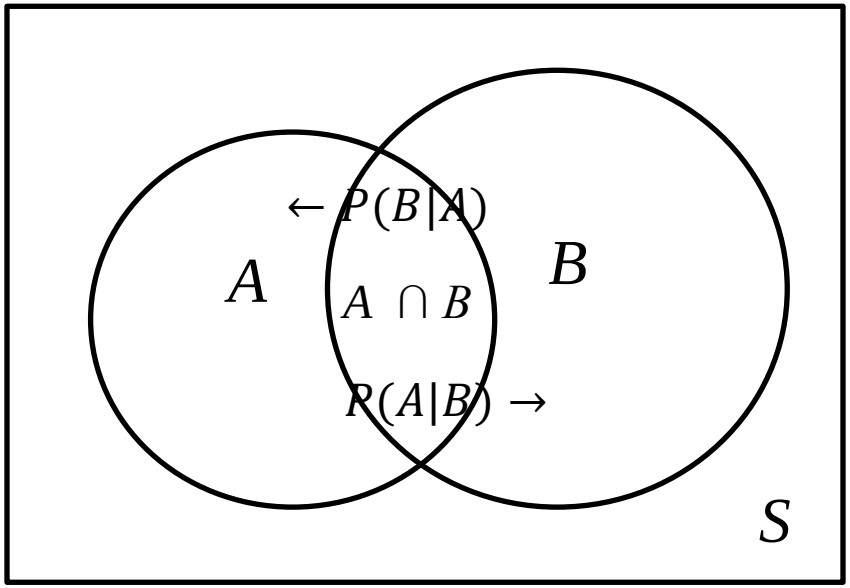
$(0.99)^2$

(b) probability of 1 test per week for 2 years all receiving accurate readings

$(0.99)^{104} \approx 0.35$ (!) (65% chance that 1 or more of the 104 tests fail)

Theorem 3.8 shows that P(A|B) and P(B|A) are related, specifically:

$$P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)} \qquad \text{(B)}$$



Remember,
P(A|B) is the ratio of the probability of event A∩B to the probability of event A
and
P(B|A) is the ratio of the probability of event A∩B to the probability of event B

Therefore to go from P(A|B) to P(B|A) one has to apply a correction, by multiplying and dividing respectively, by the probability of B and the probability of A.

In the above Figure probabilities are represented by area. P(B|A) is larger than P(A|B) by the area fraction P(B)/P(A)

Example: each year about 1/1000 people will develop lung cancer. Suppose a diagnostic method is 99.9 percent accurate (if you have cancer, it will be 99.9 Percent being diagnosed as positive, if you don't have cancer, it will be 0.1 percent Being diagnosed as positive). If you are diagnosed as positive for lunge Cancer, what is the probability that you really have cancer?

Solution: Your being not have lung cancer is A, your being diagnosed positive is B. If you are diagnosed positive, what is the probability of being healthy? That is

$$P(A/B) = \frac{P(A)}{P(B)} P(B/A)$$

P(A) = 0.999, P(B/A) = 0.001, P(B) = ?

P(B) = P(B/A)P(A) + P(B/A')P(A') = 0.001*0.999 + 0.999*0.001 = 0.001998

Substituting into the calculation:

P(A/B) = 0.999*0.001/0.0019 = 0.5

COMPOUND PROBABILITY

Probability of occurrence of any two events A and B simultaneously is the product of the unconditional probability of the occurrence of B and the probability of occurrence of the event A under the condition that B has already occurred. Symbolically,

$$P(A \cap B) = P(B) \cdot P(A/B)$$

Proof: Let there be n mutually exclusive, exhaustive, and equally likely events, out of which m cases are favourable to the event B. Out of these m cases, the event A has occurred m_1 times.

Therefore, $P(A \cap B) = \frac{m_1}{n}$, $P(B) = \frac{m}{n}$, $P(A/B) = \frac{m_1}{m}$

Thus, $P(A \cap B) = \frac{m_1}{n} = \frac{m}{n} \times \frac{m_1}{m} = P(B) \cdot P(A/B)$

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Example...

An urn has 6 black and 4 white marbles. Two of them are drawn without replacement. Find the probability of getting a black and a white marble.

Solution: Let S be the sample space. Then $n(S) = 10$

x be the event of drawing a black marble, $n(x) = 6$

y be the event of drawing a white marble, $n(y) = 4$

Now, $P(x) = \frac{6}{10} = \frac{3}{5}$, $P(y) = \frac{4}{10} = \frac{2}{5}$

$P(x/y)$ = Probability of getting white marble after drawing black $= \frac{4}{9}$

Since the events are not dependent, we have

$$P(x \cap y) = P(x).\ P(y/x) = \frac{3}{5} \times \frac{4}{9} = \frac{12}{45} = \frac{4}{15}$$

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Formulas to remember

1. De Morgan's law

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$
$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

2. ${}_nP_r = \frac{n!}{(n-r)!}$ $\binom{n}{r} = \frac{n!}{(n-r)!r!}$
3. $P(A) = \frac{N(A)}{N(S)}$
4. $N(A \cup B) = N(A) + N(B) - N(A \cap B)$
5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
6. $P(A) = 1 - P(\bar{A})$
7. For independent events A and B: $P(A \cap B) = P(A)P(B)$
8. Conditional probability: $P(A/B) = \frac{P(A \cap B)}{P(B)}$
9. If A and B are independent, $P(A/B) = P(A)$, $P(B/A) = P(B)$
10. Theorem 3.8 (bridge theorem)

$$P(A)P(B/A) = P(A \cap B) = P(B)P(A/B)$$

11. Total probability

$$P(A) = \sum_{i=1}^{n} P(B_i)P(A/B_i)$$

B_i's are mutually exclusive partition of the sample space.